# Optimal 2D convolutional codes

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CENTRO DE I&D EM MATEMÁTICA E APLICAÇÕES

CENTER FOR R&D IN MATHEMATICS AND APPLICATIONS

April 7, 2016

# Overview

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2. 2D convolutional codes
3. Superregular matrices
4. Optimal (1D) convolutional codes
5. MDS 2D convolutional codes

# 1D Convolutional Codes

Definition: A 1D convolutional code of rate $k/n$ $\mathcal{C}$ is a (free) $\mathbb{F}[z]$-submodule of $\mathbb{F}[z]^n$ of rank $k$. A full column rank matrix $G(z) \in \mathbb{F}[z]^{n\times k}$ is an encoder of $\mathcal{C}$ if

$$\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z]}\, G(z) \quad = \quad \{\mathbf{v}(z) = G(z)\mathbf{u}(z) \,|\, \mathbf{u}(z) \in \mathbb{F}^k[z]\}.$$

Definition: The distance of a 1D convolutional code $\mathcal{C}$ is defined as

$$\mathrm{dist}(\mathcal{C}) = \min\{\sum_{i\in\mathbb{N}} \mathrm{wt}(\mathbf{v_i}) \mid \mathbf{v}(z) = \sum_{i\in\mathbb{N}} \mathbf{v_i} z^i \in \mathcal{C} \text{ with } \mathbf{v}(z) \neq 0\}.$$

If $\mathcal{C}$ is a 1D convolutional code of rate $k/n$ and degree $\delta$, then

$$\mathrm{dist}(\mathcal{C}) \leq (n-k)(\lfloor \delta/k \rfloor + 1) + \delta + 1 \quad \text{(Generalized Singleton bound)}$$

# 2D convolutional codes

Definition: A 2D convolutional code of rate $k/n$ $\mathcal{C}$ is a free $\mathbb{F}[z_1, z_2]$-submodule of $\mathbb{F}[z_1, z_2]^n$ of rank $k$. A full column rank matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ is an encoder of $\mathcal{C}$ if

$$\begin{aligned}\mathcal{C} &= \mathrm{Im}_{\mathbb{F}[z_1,z_2]}\, G(z_1, z_2) \\ &= \{\mathbf{v}(z_1, z_2) = G(z_1, z_2)\mathbf{u}(z_1, z_2) \,|\, \mathbf{u}(z_1, z_2) \in \mathbb{F}^k[z_1, z_2]\}.\end{aligned}$$

The weight of a word $\widehat{v}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v(i,j) z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^n$ is defined as

$$\mathrm{wt}(\widehat{v}(z_1, z_2)) = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{wt}(v(i,j)),$$

where the weight of a constant vector $v(i,j)$ is the number of nonzero entries of $v(i,j)$

# 2D convolutional codes

The distance of a 2D convolutional code $\mathcal{C}$ is defined as

$$\mathrm{dist}(\mathcal{C}) = \min\{\sum_{i,j\in\mathbb{N}} \mathrm{wt}(v(i,j)) \mid \mathbf{v}(z_1,z_2) = \sum_{i,j\in\mathbb{N}} v(i,j) z_1^i z_2^j \in \mathcal{C}\backslash\{0\}\}.$$

Let $G(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ be an encoder of a 2D convolutional code $\mathcal{C}$ of rate $k/n$ with column degrees $\nu_1, \nu_2, \ldots, \nu_k$ and external degree $\delta = \nu_1 + \cdots \nu_k$. If $\nu_1 \geq \cdots \geq \nu_t \geq \nu_{t+1} = \cdots = \nu_k$ then

$$\mathrm{dist}(\mathcal{C}) \leq \frac{(\nu_k+1)(\nu_k+2)}{2} n - (k-t) + 1.$$

[Climent, Napp, Perea and Pinto, 2016]

Such a code is called an optimal 2D convolutional code. If $\nu_k$ takes the largest possible value (which is $\lfloor \delta/k \rfloor$), then there are only two different column degrees, and in this case, we have an MDS 2D convolutional code.

# 2D convolutional codes of rate $k/n$

Let $\mathcal{C}$ be a 2D convolutional code of rate $k/n$ and degree $\delta$ and let $\nu = \left\lfloor \frac{\delta}{k} \right\rfloor$. Then

$$\mathrm{dist}(\mathcal{C}) \leq \frac{(\nu+1)(\nu+2)}{2} n - k(\nu+1) + \delta + 1.$$

In [Climent, Napp, Perea and Pinto 2016] a construction of MDS 2D convolutional codes of rate $k/n$ and degree $\delta$ was obtained using circulant superregular matrices for $n \geq \frac{(\delta+1)(\delta+2)}{2}$

We construct superregular matrices which will enable us to construct MDS 2D convolutional codes of rate $k/n$ and degree $\delta$ for $n \geq k(\nu+1)$

Let $\mathbb{F}$ be a field, $A = [\mu_{i\ell}]$ be a square matrix of order $m$ over $\mathbb{F}$ and $S_m$ the symmetric group of order $m$. The determinant of $A$ is given by the Leibniz formula

$$| A |= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}.$$

Each $\mu_{i\sigma(i)}$ is a component of the term $\mu_\sigma = \mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}$.
A trivial term of the determinant is a term $\mu_\sigma$, with at least one component $\mu_{i\sigma(i)}$ equal to zero.
If $A$ is a square submatrix of a matrix $B$ with entries in $\mathbb{F}$, and all the terms of the determinant of $A$ are trivial, then $| A |$ is a *trivial minor* of $B$.

## Definition

A matrix $B$ is *superregular* if all its nontrivial minors are different from zero.

# Examples

## Full superregular

Cauchy matrices are examples of full superregular matrices (i. e. all its minors are nonzero).

## LT-superregular

$$\epsilon^5 + \epsilon^2 + 1 = 0 \;\Rightarrow\; \begin{pmatrix} 1 & & & & & & \\ \epsilon & 1 & & & & & \\ \epsilon^6 & \epsilon & 1 & & & & \\ \epsilon^9 & \epsilon^6 & \epsilon & 1 & & & \\ \epsilon^6 & \epsilon^9 & \epsilon^6 & \epsilon & 1 & & \\ \epsilon & \epsilon^6 & \epsilon^9 & \epsilon^6 & \epsilon & 1 & \\ 1 & \epsilon & \epsilon^6 & \epsilon^9 & \epsilon^6 & \epsilon & 1 \end{pmatrix} \in \mathbb{F}_{2^5}^{7\times 7}.$$

- Construction of classes of LT-superregular matrices is very difficult due to their triangular configuration.
- Only two classes exist:

[Rosenthal et al. (2006)] presented the first construction. For any $n$ there exists a prime number $p$ such that

$$\begin{pmatrix} \binom{n}{0} & & & \\ \binom{n-1}{1} & \binom{n}{0} & & \\ \vdots & \ddots & \ddots & \\ \binom{n-1}{n-1} & \cdots & \binom{n-1}{1} & \binom{n}{0} \end{pmatrix} \in \mathbb{F}_p^{n\times n}$$

Bad news: Requires a field with very large characteristic.

[A, Napp, and Pinto (2013)]. First construction, for any characteristic: Let $L, M \in \mathbb{N}$, $\alpha$ be a primitive element of a finite field $\mathbb{F}$ of characteristic $p$.

$$
\left[\begin{array}{ccc|c|ccc|ccc}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \alpha^{2^0} & \cdots & \alpha^{2^{M-1}} \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \alpha^{2^1} & \cdots & \alpha^{2^{M}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \alpha^{2^{M-1}} & \cdots & \alpha^{2^{2M-2}} \\
\hline
0 & \cdots & 0 & \cdots & \alpha^{2^0} & \cdots & \alpha^{2^{M-1}} & \alpha^{2^{M}} & \cdots & \alpha^{2^{2M-1}} \\
0 & \cdots & 0 & \cdots & \alpha^{2^1} & \cdots & \alpha^{2^{M}} & \alpha^{2^{M+1}} & \cdots & \alpha^{2^{2M}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \alpha^{2^{M-1}} & \cdots & \alpha^{2^{2M-2}} & \alpha^{2^{2M-1}} & \cdots & \alpha^{2^{3M-2}} \\
\hline
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hline
\alpha^{2^0} & \cdots & \alpha^{2^{M-1}} & \cdots & \alpha^{2^{M(L-1)}} & \cdots & \alpha^{2^{ML-1}} & \alpha^{2^{ML}} & \cdots & \alpha^{2^{M(L+1)-1}} \\
\alpha^{2^1} & \cdots & \alpha^{2^{M}} & \cdots & \alpha^{2^{M(L-1)+1}} & \cdots & \alpha^{2^{ML}} & \alpha^{2^{ML+1}} & \cdots & \alpha^{2^{M(L+1)}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha^{2^{M-1}} & \cdots & \alpha^{2^{2M-2}} & \cdots & \alpha^{2^{ML-1}} & \cdots & \alpha^{2^{M(L+1)-2}} & \alpha^{2^{M(L+1)-1}} & \cdots & \alpha^{2^{M(L+2)-2}}
\end{array}\right].
$$

is LT-superregular by blocks. $|\mathbb{F}|$ is very large. Can be used in Network Coding [Mahmood, Badr, Khisti, 2015].

# Superregular matrices

## Theorem [A. Napp, Pinto 2016]

Let $\mathbb{F}$ be a field and $a, b \in \mathbb{N}$, such that $a \geq b$ and $B \in \mathbb{F}^{a \times b}$. Suppose that $u = [u_i] \in \mathbb{F}^{b \times 1}$ is a column matrix such that $u_i \neq 0$ for all $1 \leq i \leq b$. If $B$ is a superregular matrix and every row of $B$ has at least one nonzero entry then

$$\mathrm{wt}(Bu) \geq a - b + 1.$$

## Idea

If the weight is smaller than $a - b + 1$ then there is a minor which is zero. Since $B$ is superregular, whenever a minor is zero it is trivial, so there many zeros in the matrix, by permutation of rows and columns we join them together and find a new square submatrix of $B$ of smaller size, but with the same properties.
By Fermat descent method we obtain a contradiction.

## Theorem BB [A. Napp, Pinto 2016]

Let $\alpha$ be a primitive element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$ and $B = [\nu_{i\,\ell}]$ be a matrix over $\mathbb{F}$ with the following properties

1. if $\nu_{i\,\ell} \neq 0$ then $\nu_{i\,\ell} = \alpha^{\beta_{i\,\ell}}$ for a positive integer $\beta_{i\,\ell}$;
2. If $\nu_{i\,\ell} = 0$ then $\nu_{i'\,\ell} = 0$, for any $i' > i$ or $\nu_{i\,\ell'} = 0$, for any $\ell' < \ell$;
3. if $\ell < \ell'$, $\nu_{i\ell} \neq 0$ and $\nu_{i\ell'} \neq 0$ then $2\beta_{i\,\ell} \leq \beta_{i\,\ell'}$;
4. if $i < i'$, $\nu_{i\,\ell} \neq 0$ and $\nu_{i'\,\ell} \neq 0$ then $2\beta_{i\,\ell} \leq \beta_{i'\,\ell}$.

Suppose $N$ is greater than any exponent of $\alpha$ appearing as a nontrivial term of any minor of $B$. Then $B$ is superregular.

## Theorem AA

Let $\alpha$ be a primitive element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$ and $B = [\nu_{i\,\ell}]$ be a matrix over $\mathbb{F}$ with the following properties

1. if $\nu_{i\,\ell} \neq 0$ then $\nu_{i\,\ell} = \alpha^{\beta_{i\,\ell}}$ for a positive integer $\beta_{i\,\ell}$;
2. If $\nu_{i\,\ell} = 0$ then $\nu_{i'\,\ell} = 0$, for any $i' < i$ or $\nu_{i\,\ell'} = 0$, for any $\ell' > \ell$;
3. if $\ell < \ell'$, $\nu_{i\ell} \neq 0$ and $\nu_{i\ell'} \neq 0$ then $2\beta_{i\,\ell} \leq \beta_{i\,\ell'}$;
4. if $i < i'$, $\nu_{i\,\ell} \neq 0$ and $\nu_{i'\,\ell} \neq 0$ then $2\beta_{i\,\ell} \leq \beta_{i'\,\ell}$.

Suppose $N$ is greater than any exponent of $\alpha$ appearing as a nontrivial term of any minor of $B$. Then $B$ is superregular.

# Example

Let $E = [e_{ij}]$ be the matrix

$$\begin{bmatrix} \emptyset & \emptyset & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 7 \\ 2 & \emptyset & 5 & 6 & 7 & 8 \\ \emptyset & \emptyset & 6 & 7 & \emptyset & 9 \\ \emptyset & \emptyset & 7 & 8 & \emptyset & \emptyset \end{bmatrix}$$

and $C = [c_{ij}]$ be the $6 \times 6$ matrix defined by

$$c_{ij} = \begin{cases} 0 & \text{if } e_{ij} = \emptyset \\ \alpha^{2^{e_{ij}}} & \text{elsewhere} \end{cases} .$$

The matrix $F = [f_{ij}] = [E_3\ E_4\ E_1\ E_2\ E_5\ E_6]$, where $E_i$ represents the *i-th* column of $E$, is

# Example

$$\begin{bmatrix} 2 & 3 & \emptyset & \emptyset & 4 & 5 \\ 3 & 4 & 0 & 1 & 5 & 6 \\ 4 & 5 & 1 & 2 & 6 & 7 \\ 5 & 6 & 2 & \emptyset & 7 & 8 \\ 6 & 7 & \emptyset & \emptyset & \emptyset & 9 \\ 7 & 8 & \emptyset & \emptyset & \emptyset & \emptyset \end{bmatrix}$$

and therefore $A = [a_{ij}]$, the $6 \times 6$ matrix, defined by

$a_{ij} = \begin{cases} 0 & \text{if } a_{ij} = \emptyset \\ \alpha^{2^{f_{ij}}} & \text{elsewhere} \end{cases}$ satisfies properties

(i) if $\hat{\sigma} \in S_m$ is the permutation defined by $\hat{\sigma}(i) = m - i + 1$, then $\mu_{\hat{\sigma}}$ is a nontrivial term of $|\, A \,|$.

(ii) if $\ell \geq m - i + 1$, $\ell < \ell'$, $\mu_{i\ell} \neq 0$ and $\mu_{i\ell'} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i\ell'}$;

(iii) if $\ell \geq m - i + 1$, $i < i'$, $\mu_{i\ell} \neq 0$ and $\mu_{i'\ell} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i'\ell}$.

# Example

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \alpha^{2^3} & \alpha^{2^4} & \alpha^{2^{12}} & \alpha^{2^{13}} \\ 0 & 0 & 0 & \alpha^{2^6} & \alpha^{2^7} & \alpha^{2^{15}} & \alpha^{2^{16}} \\ 0 & \alpha^{2^0} & \alpha^{2^1} & \alpha^{2^9} & \alpha^{2^{10}} & \alpha^{2^{18}} & 0 \\ 0 & \alpha^{2^3} & \alpha^{2^4} & \alpha^{2^{12}} & \alpha^{2^{13}} & \alpha^{2^{21}} & 0 \\ 0 & \alpha^{2^6} & \alpha^{2^7} & \alpha^{2^{15}} & \alpha^{2^{16}} & \alpha^{2^{24}} & 0 \\ \alpha^{2^1} & \alpha^{2^9} & \alpha^{2^{10}} & \alpha^{2^{18}} & 0 & 0 & 0 \\ \alpha^{2^4} & \alpha^{2^{12}} & \alpha^{2^{13}} & \alpha^{2^{21}} & 0 & 0 & 0 \end{bmatrix}.$$

Let $\hat{\sigma} \in S_7$ be the permutation defined by $\hat{\sigma}(i) = 8 - i$ and let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 5 & 3 & 2 & 4 & 1 \end{pmatrix}.$$

# Optimal (1D) Convolutional codes

Let $\mathcal{C}$ be a convolutional code of rate $k/n$ and different Forney indices $\nu_1 < \cdots < \nu_\ell$ with corresponding multiplicities $m_1, \ldots, m_\ell$ and

$$G(z) = \sum_{i=0}^{\nu_\ell} G_i z^i$$

a column reduced encoder of $\mathcal{C}$ with column degrees in nondecreasing order. Consider a nonzero codeword $v(z) = G(z)u(z)$ with $u(z) \in \mathbb{F}[z]^k$. Write

$$u(z) = \sum_{i=0}^{\epsilon} u_i z^i \quad \text{and} \quad v(z) = \sum_{i=0}^{\nu_\ell+\epsilon} v_i z^i,$$

A convolutional code of rate $k/n$ with different Forney indices $\nu_1 < \cdots < \nu_\ell$ and with corresponding multiplicities $m_1, \ldots, m_\ell$ and distance $n(\nu_1 + 1) - m_1 + 1$ is said to be an *optimal* $(n, k, \nu_1, m_1)$ convolutional code.

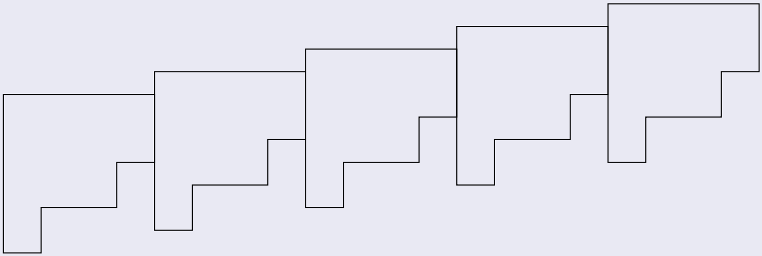
# Optimal (1D) Convolutional codes

If $G(z)$ is such that the matrices $\mathcal{G}(\epsilon)$ are superregular, then $\mathcal{C}$ is an *optimal* $(n, k, \nu_1, m_1)$ *convolutional code* [A., Napp and Pinto, 2016].

$$\mathcal{G}(\epsilon) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & G_0 \\ 0 & 0 & \cdots & 0 & G_0 & G_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & G_{\nu_\ell-2} & G_{\nu_\ell-1} & G_{\nu_\ell} \\ 0 & 0 & \cdots & G_{\nu_\ell-1} & G_{\nu_\ell} & 0 \\ 0 & 0 & \cdots & G_{\nu_\ell} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ G_0 & G_1 & \cdots & 0 & 0 & 0 \\ G_1 & G_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ G_{\nu_\ell-1} & G_{\nu_\ell} & \cdots & 0 & 0 & 0 \\ G_{\nu_\ell} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{F}^{n(\nu_\ell+\epsilon+1)\times k(\epsilon+1)}.$$

# Optimal (1D) Convolutional codes

## Structure



# Optimal (1D) Convolutional codes

## Theorem

Let $G(z) = \sum_{i \geq 0} G_i z^i \in \mathbb{F}[z]^{n \times k}$ be a matrix with column degrees $\nu_1 < \cdots < \nu_\ell$ with multiplicities $m_1, \ldots, m_\ell$, respectively, and such that all entries of the last $m_j + \cdots + m_\ell$ columns of $G_i$ are nonzero for $i \leq \nu_j$, $j = 1, \ldots, \ell$. Suppose that $\mathcal{G}(\epsilon_0)$, is superregular for

$$\epsilon_0 = \left\lceil \frac{n(\nu_1 + 1) - m_1}{n - k} \right\rceil - 1.$$

Then $G(z)$ is column reduced and $\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z]} G(z)$ is an optimal $(n, k, \nu_1, m_1)$ convolutional code, i.e. the distance of the code is equal to $n(\nu_1 + 1) - m_1 + 1$.

Whether this bound was optimal or not was left as an open question [Gluersing-Luerssen, Rosenthal and Smarandache, 2006].

# MDS 2D Convolutional codes

Our construction of a MDS 2D convolutional code will be based on constructions of optimal 1D convolutional codes and superregular matrices. Given an encoder

$$\widehat{G}(z_1, z_2) = \sum_{0 \leq a+b \leq \nu+1} G_{a,b} z_1^a z_2^b \in \mathbb{F}[z_1, z_2]^{n \times k}$$

of a 2D convolutional code $\mathcal{C}$, we can write

$$\widehat{G}(z_1, z_2) = \sum_{j=0}^{\nu+1} G_j(z_1) z_2^j.$$

$$\widehat{u}(z_1, z_2) = \sum_{j=0}^{\ell} \widehat{u}_j(z_1) z_2^j \quad \text{and} \quad \widehat{v}(z_1, z_2) = \sum_{j=0}^{\nu+1+\ell} \widehat{v}_j(z_1) z_2^j$$

# MDS 2D Convolutional codes

We will also consider $\widehat{u}_0(z_1) \neq 0$, $\widehat{u}_\ell(z_1) \neq 0$ and $\widehat{u}_a(z_1) = 0$ if $a > \ell$ or if $a < 0$. Therefore,

1. If $0 \leq s \leq \nu$,
$$\widehat{v}_s(z_1) = \sum_{j=0}^{s} G_j(z_1)\widehat{u}_{s-j}(z_1);$$
2. If $\nu + 1 \leq s \leq \ell$,
$$\widehat{v}_s(z_1) = \sum_{j=0}^{\nu+1} G_j(z_1)\widehat{u}_{s-j}(z_1);$$
3. If $\ell + 1 \leq s \leq \ell + 1 + \nu$,
$$\widehat{v}_s(z_1) = \sum_{j=s-\ell}^{\nu+1} G_j(z_1)\widehat{u}_{s-j}(z_1).$$

# Optimal (1D) Convolutional codes for 2D MDS codes

If $\ell \neq 0$, for each $s \in \{0, 1, 2, \dots, \nu\} \cup \{\ell + \nu\}$, we may regard $\widehat{v}_s(z_1)$ as codewords of a 1D convolutional code $\mathcal{C}_s$ with the following characteristics: If $0 \leq s \leq \nu$, $\mathcal{C}_s$ is a 1D convolutional code of rate $\frac{(s'+1)k}{n}$

$$\widehat{\mathcal{G}}_s(z_1) = [G_0(z_1) \quad G_1(z_1) \quad \cdots \quad G_{s'}(z_1)] \in \mathbb{F}[z_1]^{n \times (s'+1)k},$$

with Forney indices $\nu_i = \nu + i - 1 - s$, for $i \in \{1, 2, \dots, s+2\}$ and the multiplicity of $\nu_1$ is $k(\nu+1) - \delta$, the multiplicity of $\nu_i$, for any $i \in \{2, \dots, s+1\}$ is $k$ and the multiplicity of $\nu_{s+2}$ is $\delta - k\nu$ (notice that, if $k \mid \delta$, we will have $s+1$ Forney indices all with multiplicity $k$).

# Optimal (1D) Convolutional codes for 2D MDS codes

the encoder of $\mathcal{C}_s$ is the matrix

$$\begin{aligned} \widehat{\mathcal{G}}_s(z_1) &= [G_0(z_1) \quad G_1(z_1) \quad \cdots \quad G_{s'}(z_1)] \\ &= G_0^{(s)} + G_1^{(s)} z_1 + \cdots + G_{\nu+1}^{(s)} z_1^{\nu+1}, \end{aligned}$$

where, for $0 \leq i \leq \nu + 1$

$$G_i^{(s)} = \begin{bmatrix} G_{i,0} & G_{i,1} & \cdots & G_{i,s'} \end{bmatrix}.$$

# Superregular?

$$\mathcal{G}(\epsilon, s) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & G_0^{(s)} \\ 0 & 0 & \cdots & 0 & G_0^{(s)} & G_1^{(s)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & G_{\nu-1}^{(s)} & G_\nu^{(s)} & G_{\nu+1}^{(s)} \\ 0 & 0 & \cdots & G_\nu^{(s)} & G_{\nu+1}^{(s)} & 0 \\ 0 & 0 & \cdots & G_{\nu+1}^{(s)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ G_0^{(s)} & G_1^{(s)} & \cdots & 0 & 0 & 0 \\ G_1^{(s)} & G_2^{(s)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ G_\nu^{(s)} & G_{\nu+1}^{(s)} & \cdots & 0 & 0 & 0 \\ G_{\nu+1}^{(s)} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{F}^{n(\nu+\epsilon+2)\times k(\epsilon+1)}.$$

# Optimal (1D) Convolutional codes for 2D MDS codes

If $s = \nu + \ell$, then $\mathcal{C}_{\nu+\ell}$ is a 1D convolutional code of rate $\frac{\delta - k(\nu-1)}{n}$ whose encoder is the matrix

$$\widehat{\mathcal{G}}_{\nu+\ell}(z_1) = \begin{bmatrix} G_\nu(z_1) & \widetilde{G}_{\nu+1}(z_1) \end{bmatrix} \in \mathbb{F}[z_1]^{n\times(\delta - k(\nu-1))}$$

where $\widetilde{G}_{\nu+1}(z_1)$ is the submatrix of $G_{\nu+1}(z_1)$ formed by its first $\delta - k\nu$ columns. If $k \mid \delta$ then $\widehat{\mathcal{G}}_{\nu+\ell}(z_1) = G_\nu(z_1)$. If $k \nmid \delta$ the Forney indices are $\nu_1 = 0$ with multiplicity $k$ and $\nu_2 = 1$, with multiplicity $\delta - k\nu$. If $k \mid \delta$ there is only one Forney index, $n_1 = 1$, whose multiplicity is $k$.

# Optimal (1D) Convolutional codes for 2D MDS codes

$$\begin{aligned}\widehat{\mathcal{G}}_{\nu+\ell}(z_1) &= \begin{bmatrix} G_\nu(z_1) & \widetilde{G}_{\nu+1}(z_1) \end{bmatrix} \\ &= G_0^{(\nu+\ell)} + G_1^{(\nu+\ell)} z_1,\end{aligned}$$

where, if we represent by $\widetilde{G}_{0,\nu+1}$ the matrix formed by the first $\delta - k\nu$ columns of $G0, \nu + 1$,

$$G_0^{(\nu+\ell)} = \begin{bmatrix} G_{0,\nu} & \widetilde{G}_{0,\nu+1} \end{bmatrix}$$

and

$$G_1^{(\nu+\ell)} = [G_{1,\nu} \quad 0] .$$

# Superregular?

$$\mathcal{G}(\epsilon, \nu+\ell) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & G_0^{(\nu+\ell)} \\ 0 & 0 & \cdots & 0 & G_0^{(\nu+\ell)} & G_1^{(\nu+\ell)} \\ 0 & 0 & \cdots & G_0^{(\nu+\ell)} & G_1^{(\nu+\ell)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ G_0^{(s)} & G_1^{(s)} & \cdots & 0 & 0 & 0 \\ G_1^{(s)} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

# And if $\ell = 0$?

If $\ell = 0$, instead of having the polynomials of first in terms of $z_2$, we will consider them first in terms of $z_1$, i. e.

$$\begin{aligned}\widehat{u}(z_1, z_2) &= \widehat{u}_0(z_1) \\ &= u_{0,0} + u_{1,0}z_1 + u_{2,0}z_1^2 + \cdots + u_{\ell_0,0}z_1^{\ell_0},\end{aligned}$$

$$\widehat{G}(z_1, z_2) = \sum_{i=0}^{\nu+1} \widehat{G}_i(z_2) z_1^i.$$

We write, for $i \in \{0, 1, \ldots, \nu + 1\}$,

$$\widehat{G}_i(z_2) = \sum_{j=0}^{\nu+1} G_{i,j} z_2^j$$

where, if $i + j > \nu + 1$, $G_{i,j}$ is a $n \times k$ null matrix.

# And if $\ell = 0$?

Consider also

$$\bar{G}_0 = \begin{bmatrix} G_{0,0} \\ G_{0,1} \\ \vdots \\ G_{0,\nu} \\ G_{0,\nu+1} \end{bmatrix} \quad \bar{G}_1 = \begin{bmatrix} G_{1,0} \\ G_{1,1} \\ \vdots \\ G_{1,\nu-1} \\ G_{1,\nu} \\ O_1 \end{bmatrix} \quad \bar{G}_i = \begin{bmatrix} G_{i,0} \\ G_{i,1} \\ \vdots \\ G_{i,\nu-i} \\ G_{i,\nu+1-i} \\ O_1 \\ \vdots \\ O_i \end{bmatrix}$$

for each $i \in \{2, 3, \ldots, \nu+1\}$, and where each matrix $O_j$, is a null column $n \times k$ matrix. If $k \nmid \delta$ and $i \in \{0, 1, \ldots, \nu+1\}$ the last $k(\nu+1) - \delta$ columns of $G_{i,\nu+1-i}$ are also null columns.

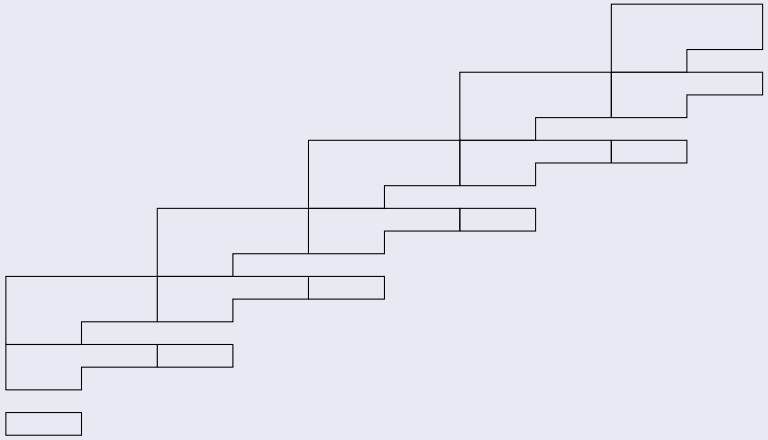
# And if $\ell = 0$?

## Superregular?

$$\mathcal{G}(n,k,\delta,\ell_0) =$$

$$\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \bar{G}_0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \bar{G}_0 & \bar{G}_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \bar{G}_0 & \cdots & \bar{G}_\nu & \bar{G}_{\nu+1} \\
0 & 0 & \cdots & 0 & 0 & \cdots & \bar{G}_0 & \bar{G}_1 & \cdots & \bar{G}_{\nu+1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{G}_0 & \bar{G}_1 & \cdots & \bar{G}_\nu & \bar{G}_{\nu+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\bar{G}_1 & \bar{G}_2 & \cdots & \bar{G}_{\nu+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{G}_\nu & \bar{G}_{\nu+1} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\bar{G}_{\nu+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.$$

# The zero structure of $\mathcal{G}(n, 4, 6, 4)$



# The entries of these superregular matrices

Let $\alpha$ be a primitive element of the finite field $\mathbb{F} = \mathbb{F}_{p^N}$ and for $0 \leq a, b \leq \nu + 1$, define $G_{a,b} = \left[ g_{i,j}^{(a,b)} \right] \in \mathbb{F}^{n \times k}$ by

$$g_{i,j}^{(a,b)} = \begin{cases} \alpha^{2^{(a(\nu+2)+b)n+i+j-2}} & \text{if} \quad 0 \leq a+b \leq \nu \\ \alpha^{2^{(a(\nu+2)+b)n+i+j-2}} & \text{if} \quad a+b = \nu+1 \quad \text{and} \quad j \leq \delta - k\nu \\ 0 & \text{if} \quad a+b = \nu+1 \quad \text{and} \quad j > \delta - k\nu \\ 0 & \text{if} \quad a+b > \nu+1. \end{cases}$$

# MDS 2D convolutional codes

**Theorem**

Let $N$ be sufficiently large, $\delta \geq 0$, $k \geq 1$, $\nu = \left\lfloor \frac{\delta}{k} \right\rfloor$ and $n \geq k(\nu + 1)$. Consider $\widehat{G}(z_1, z_2)$ with $G(a, b)$ defined above. Then $\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z_1,z_2]} \widehat{G}(z_1, z_2)$ is a 2D MDS convolutional code of rate $k/n$ and degree $\delta$.