Constant-Dimension Codes Exceeding the LMRD Code Bound

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Outline

Outline

Subspace Coding

The constant-dimension case

Definition

A q-ary $(v, M, d; k)$ *(constant-dimension) subspace code* is a set $\mathcal{C}$ of k-dimensional subspaces of a v-dimensional vector space over $\mathbb{F}_q$ with size $\#\mathcal{C} = M$ and minimum subspace distance $\mathrm{d_s}(\mathcal{C}) := \min\{\mathrm{d_s}(X, Y); X, Y \in \mathcal{C}, X \neq Y\} = d$.

Subspace metric

$\mathrm{d_s}(X, Y) = \dim(X + Y) - \dim(X \cap Y) = 2k - 2\dim(X \cap Y)$

Geometric meaning

$d = 2\delta \in 2\mathbb{Z}$, and $t = k - \delta + 1$ is the smallest positive integer such that any t-dimensional subspace of V (or $t-1$-flat of $\mathrm{PG}(V) \cong \mathrm{PG}(v-1, \mathbb{F}_q)$) is covered by/contained in/incident with at most one member of $\mathcal{C}$.

Main Problem

For a given prime power $q > 1$ and given positive integers v, δ, k with $2 \leq \delta \leq k \leq v/2$ determine the maximum size $M = \mathrm{A}_q(v, 2\delta; k)$ of a q-ary $(v, M, 2\delta; k)$ subspace code.

The Case $k = 3$, $d = 4$

Plane subspace codes

The "easiest" "nontrivial" case

A q-ary $(v, M, 4; 3)$ subspace code is a set of M distinct planes in $\mathrm{PG}(V) \cong \mathrm{PG}(v-1, \mathbb{F}_q)$ mutually intersecting in at most a point (covering every line at most once).

Known exact results

1. $\mathrm{A}_q(5, 4; 3) = q^3 + 1$ ($\triangleq$ max. partial line spreads in $\mathrm{PG}(4, \mathbb{F}_q)$)
2. $\mathrm{A}_2(6, 4; 3) = 77$ (5 isomorphism types)
3. $\mathrm{A}_2(13, 4; 3) = 1\,597\,245$ (many isomorphism types)

The $(13, 1\,597\,245, 4; 3)$ codes in Case (3) form an exact line cover in $\mathrm{PG}(12, \mathbb{F}_2)$ (2-analog of a Steiner triple system on 13 points) and are invariant under the normalizer of a Singer group of $\mathrm{PG}(12, \mathbb{F}_2)$, which has order $(2^{13} - 1) \times 13 = 106\,483$.

It is not known whether an exact line cover (consisting of planes) in $\mathrm{PG}(6, \mathbb{F}_2)$ (2-analog of the Fano plane) or in $\mathrm{PG}(8, \mathbb{F}_2)$ (2-analog of the affine plane of order 3) exists.

Known Upper Bounds for $\mathrm{A}_q(v,4;3)$

Packing bound

$$\#\mathcal{C} \leq \frac{\text{total no. of lines}}{\text{no. of lines in a plane}} = \frac{(q^v-1)(q^{v-1}-1)}{(q^3-1)(q^2-1)}$$

with equality iff $\mathcal{C}$ forms an exact line cover (q-analog of a Steiner triple system on v points).

Best known upper bound

$$\#\mathcal{C} \leq \begin{cases} \left\lfloor \frac{(q^v-1)(q^{v-1}-1)}{(q^3-1)(q^2-1)} \right\rfloor & \text{if } v \equiv 1 \pmod 2, \\ \left\lfloor \frac{q^v-1}{q^3-1}\left(\frac{q^{v-1}-q}{q^2-1} - q + 1\right) \right\rfloor & \text{if } v \equiv 0 \pmod 2, \end{cases}$$

$$= \begin{cases} (q^3+1)^2 & \text{if } v = 6, \\ q^8+q^6+q^5+q^4+q^3+q^2+1 & \text{if } v = 7, \\ q^{2v-6}+q^{2v-8}+q^{2v-9}+\cdots & \text{if } v \geq 8. \end{cases}$$

A necessary condition for the existence of an exact cover is $v \equiv 1,3 \pmod 6$ (independently of q).

Known Lower Bounds for $\mathrm{A}_q(v,4;3)$

Mostly arising from constructions

- $\mathrm{A}_q(6,4;3) \geq q^6 + 2q^2 + 2q + 1$ for $q \geq 3$;
- $\mathrm{A}_2(7,4;3) \geq 333$, $\mathrm{A}_3(7,4;3) \geq 6977$, and $\mathrm{A}_q(7,4;3) \geq q^8 + q^5 + q^4 + q^2 - q$ for general q;
- $\mathrm{A}_q(v,4;3) \geq q^{2v-6} + \begin{bmatrix} v-3 \\ 2 \end{bmatrix}_q = q^{2v-6} + q^{2v-10} + \cdots$ for q large enough (*LMRD code bound*, constructive);
- $\mathrm{A}_q(v,4;3) \sim \frac{(q^v-1)(q^{v-1}-1)}{(q^3-1)(q^2-1)}$ for v large enough (packing bound, non-constructive).

The binary case

v	6	7	8	9	10	11
LMRD	71	291	1179	4747	19051	76331
EA+Ext	**77**	**329**	**1259**	**5014**	**20517**	**79306**
best known	77	333	1326	5986	23870	97526
upper bound	77	381	1493	6205	24698	99718

EA+Ext: Expurgation-Augmentation plus further extension by planes meeting the special flat S in a line

The Echelon-Ferrers Construction

The Echelon-Ferrers Multilevel Construction and its refinements (T. Etzion, N. Silberstein, J. Rosenthal, A. Horlemann-Trautmann) provides the best known lower bound for subspace codes with general parameters.

Idea (for the plane case)

Take

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & * & \dots & * \\ 0 & 1 & 0 & * & \dots & * \\ 0 & 0 & 1 & * & \dots & * \end{pmatrix} \uplus \begin{pmatrix} 1 & * & * & 0 & 0 & * & \dots & * \\ 0 & 0 & 0 & 1 & 0 & * & \dots & * \\ 0 & 0 & 0 & 0 & 1 & * & \dots & * \end{pmatrix} \uplus \cdots$$

with the maximum number of planes from each Schubert cell.
$\Longrightarrow \#\mathcal{C} = 2^{2(v-3)} + 2^{2(v-5)} + \cdots$ in the binary case.

LMRD code bound

$$\#\mathcal{C} \leq 2^{2(v-3)} + \begin{bmatrix} v-3 \\ 2 \end{bmatrix}_2$$

Outline

Main Theorem (Ai-H.-Liu, 2016)

(i) *For* $v \equiv 7 \pmod 8$*, there exists a* Σ_v*-invariant* $(v, M, 4; 3)_2$ *subspace code with*

$$M \geq 2^{2(v-3)} + \frac{9}{8}\begin{bmatrix} v-3 \\ 2 \end{bmatrix}_2,$$

and consequently we have $\mathrm{A}_2(v, 4; 3) \geq 2^{2(v-3)} + \frac{9}{8}\left[\begin{smallmatrix} v-3 \\ 2 \end{smallmatrix}\right]_2$ *in this case.*

(ii) *For* $v \equiv 3 \pmod 8$*,* $v \geq 11$*, there exists a* Σ_v*-invariant* $(v, M, 4; 3)_2$ *subspace code with*

$$M \geq 2^{2(v-3)} + \frac{81}{64}\begin{bmatrix} v-3 \\ 2 \end{bmatrix}_2,$$

and consequently we have $\mathrm{A}_2(v, 4; 3) \geq 2^{2(v-3)} + \frac{81}{64}\left[\begin{smallmatrix} v-3 \\ 2 \end{smallmatrix}\right]_2$ *in this case.*

Maximum Net Gain Computations

v	n	#G-orbits	N_1	$(N_1)_{\mathrm{LMRD}}$	$\#\mathcal{C}$	W
7	4	1	3	2.33	$2^8 + 45$	$\langle 1, \alpha, \alpha^2 \rangle$
8	5	1	3	5.00	$2^{10} + 93$	$\langle 1, \alpha, \alpha^2 \rangle$
9	6	7	12	10.33	$2^{12} + 756$	$\langle 1, \alpha, \alpha^2 \rangle$
10	7	15	20	21.00	$2^{14} + 2540$	$\langle 1, \alpha, \alpha^{22} \rangle$
11	8	53	54	42.33	$2^{16} + 13770$	$\langle 1, \alpha^{17}, \alpha^{34} \rangle$
12	9	177	93	85.00	$2^{18} + 47523$	$\langle 1, \alpha^3, \alpha^{71} \rangle$
13	10	633	234	170.33	$2^{20} + 239382$	$\langle 1, \alpha, \alpha^{49} \rangle$
14	11	513	379	341.00	$2^{22} + 775813$	$\langle 1, \alpha^3, \alpha^{419} \rangle$
15	12	34	924	682.33	$2^{24} + 3783708$	$\langle 1, \alpha^{195}, \alpha^{1170} \rangle$
16	13	240	1526	1365.00	$2^{26} + 12499466$	$\langle 1, \alpha^{25}, \alpha^{1208} \rangle$

N_1 local max. net gain of the EA method

$(N_1)_{\mathrm{LMRD}}$ local max. net gain equivalent of the LMRD code bound

Outline

Coordinate-Free Represenation

From now on we restrict ourselves to $k = 3$, $d = 4$, $q = 2$.

Ambient space

$V = W \times \mathbb{F}_{2^n}$, where $n = v - 3$ and W is a 3-dimensional $\mathbb{F}_2$-subspace of $\mathbb{F}_{2^n}$ (plane of $\mathrm{PG}(n-1, \mathbb{F}_2)$)

Gabidulin MRD codes

$\mathcal{G} = \{x \mapsto a_0x + a_1x^2; a_0, a_1 \in \mathbb{F}_{2^n}\} \subset \mathrm{Hom}(W, \mathbb{F}_{2^n})$
(for $n \geq 6$ this definition depends on the choice of W)

Lifted Gabidulin LMRD codes

$\mathcal{L}$ = set of all graphs (in the sense of Real Analysis) Γ_f, $f \in \mathcal{G}$; i.e.,

$$G(a_0, a_1) = \left\{(x, a_0x + a_1x^2); x \in W\right\} \subset W \times \mathbb{F}_{2^n}$$

Lines covered by $G(a_0, a_1)$

These have the form Γ_g, where g is the restriction of $a_0x + a_1x^2$ to a 2-dimensional subspace $Z \subset W$, and are disjoint from the special flat

$$S = \{0\} \times \mathbb{F}_{2^n} \cong \mathrm{PG}(n-1, \mathbb{F}_2).$$

The LMRD Code Bound

Valid for any subspace code $\mathcal{C}$ containing an LMRD code

Observation

The planes in $\mathcal{L}$ (more generally, the planes in any lifted MRD code with the same parameters as $\mathcal{G}$) form an exact cover of the set of lines of $\mathrm{PG}(v-1,\mathbb{F}_2)$ disjoint from S.

$\Longrightarrow$ No plane meeting S in a point can be added to $\mathcal{L}$ without decreasing the minimum subspace distance (since such planes contain lines disjoint from S, hence leading to a multiple cover of some line).

$$\Longrightarrow \#\mathcal{C} \leq \#\mathcal{L} + \text{no. of lines in } S = 2^{2v-6} + \begin{bmatrix} v-3 \\ 2 \end{bmatrix}_2$$

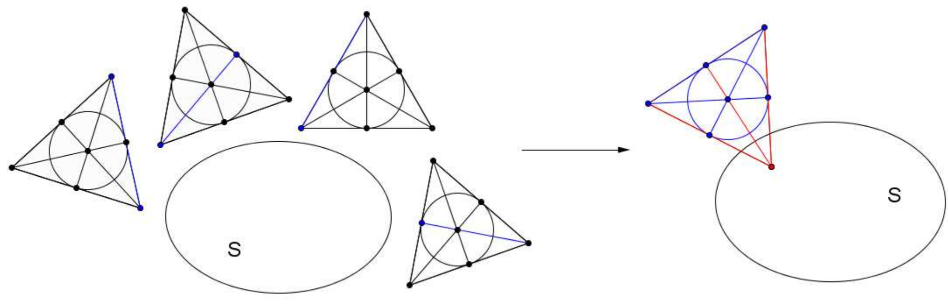
for any subspace code $\mathcal{C} \supseteq \mathcal{L}$.

Can the bound be reached?

For this the lines $L \subset S$ must be matched to planes $E \supset L$ in such a way that planes meeting in S (i.e., the corresponding lines meet) have no point outside S in common.

The answer is yes for $v \leq 11$ and probably in general.

Outline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Expurgation-Augmentation

The basic idea

Removing M_1 planes from $\mathcal{L}$ ("expurgating" $\mathcal{L}$) "frees" $7M_1$ lines disjoint from the special flat S. It is at least conceivable that the free lines can be rearranged, 4 lines at a time, into $7M_1/4$ new planes meeting S in a point.

Adding these planes to the expurgated LMRD code ("augmenting" the code) then produces a new subspace code $\mathcal{C}$ of size

$$\#\mathcal{C} = \#\mathcal{L} + 3M_1/4 > \#\mathcal{L}.$$

If we are "lucky", the new planes do not introduce a multiple cover of some line meeting S in a point.

If we are even more "lucky", the additional number of planes meeting S in a line that can be added to the code does not decrease (or decreases only slightly).

$\Longrightarrow \mathcal{C}$ improves on $\mathcal{L}$.

After some further work

There exists a distinguished 3-dimensional subspace $\mathcal{T} \subset \mathcal{G}$, viz.

$$\mathcal{T} = \{wx^2 + w^2x; w \in W\},$$

such that the corresponding 8 planes Γ_f, $f \in \mathcal{T}$, have the desired property.

The 14 new planes obtained by rearranging the 8×7 lines in Γ_f are

$$E = E(Z, P, g) = \bigl\{(x, g(x) + y); x \in Z, y \in P\bigr\},$$

where $Z = \langle a, b\rangle \subset W$ is 2-dimensional (7 choices), $g(x) = cx^2 + c^2x$ with $c \in W/Z$ (2 choices) and $P = \mathbb{F}_2(ab^2 + a^2b)$ (the intersection point of E and S).

Net gain: $\quad 14 - 8 = 6$ planes

Example ($v = 6$)

One of the five optimal $(6, 77, 4; 3)$ codes can be constructed in this way without using a computer. In this case $V = \mathbb{F}_8 \times \mathbb{F}_8$ (i.e. $W = \mathbb{F}_8$) and $\mathcal{T} = \{wx^2 + w^2x; w \in \mathbb{F}_8\}$.

Refinements for $v = 7$

For $v = 7$ the ambient space can be taken as $V = W \times \mathbb{F}_{16}$, with W the trace-zero subspace of $\mathbb{F}_{16}$.

1. Remove several additive cosets of $\mathcal{T}$ in $\mathcal{G}$ (maximum 2 cosets, netgain 12 planes).
2. Remove pairwise disjoint "rotated" cosets $r(\mathcal{T} + f)$, $f \in \mathcal{G}$, $r \in \mathbb{F}_{16}^{\times}$ (maximum 4 cosets, netgain 24 planes)
3. Remove all $\#\mathbb{F}_{16}^{\times} = 15$ rotations of the special coset $\mathcal{T} + cx^2 + c^2x$, $\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}(c) = 1$, but drop the requirement of exact rearrangement of the free lines (net gain $15 \times 11 - 15 \times 8 = 45$ planes)

Why is Method (3) so much better?

- The expurgated code is invariant under the group Σ_v of all collinations $(x, y) \mapsto (x, ry)$, $r \in \mathbb{F}_{16}^{\times}$ (acting as a Singer group on $\mathrm{PG}(S) \cong \mathrm{PG}(3, \mathbb{F}_2)$). $\implies$ Simplification
- Surprisingly (at that time) as much as 11 out of 14 candidate new planes could be added through each point of S.

A Strange Invariant

determining the collision graph at a point of S

Collision graph

Vertices: the 14 new planes E through a fixed point of S, say $P_1 = \mathbb{F}_2(0,1) \in W \times \mathbb{F}_{16}$.
Edges: E_1 and E_2 are adjacent if they have a line through P_1 (or a point outside S) in common.

In the case $v = 7$ the graph turned out to consist of a K_4 and 10 isolated vertices ($\Longrightarrow$ independence number 11).

δ-invariant (last Dickson invariant)

Represent $\mathrm{PG}(n-1,\mathbb{F}_q)$ as $\mathrm{PG}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. For any $\mathbb{F}_q$-subspace U define the point $\delta(U)$ as the product of all points in U.

Note that for a line $Z = \langle a,b\rangle = \{a,b,a+b\} \in \mathrm{PG}(n-1,\mathbb{F}_2)$ we have $\delta(Z) = ab(a+b) = ab^2 + a^2b = \begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$.

σ-invariant

For a plane E in $\mathrm{PG}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ intersecting W in a line Z define $\sigma(E) = \delta(E)/\delta(Z)^{q+1}$.

Theorem

1. *The 14 new planes through P_1 have the form $E(Z, P_1, g)$ with $Z = \langle a, b\rangle \subset W = \langle a, b, c\rangle \subset \mathbb{F}_{16} = \langle a, b, c, d\rangle$ and*

$$g(x) = \frac{(d+\mu c)x^2 + (d+\mu c)^2 x}{ab^2 + a^2 b}, \quad \mu \in \mathbb{F}_2.$$

   $E(Z, P_1, g) \mapsto Z + \mathbb{F}_q(d + \mu c)$ gives a parametrization of these new planes by the 14 planes $E \neq W$ in $\mathrm{PG}(S) \cong \mathrm{PG}(3, \mathbb{F}_2)$.

2. *Two new planes $E(Z, P_1, g)$, $E(Z', P_1, g')$ collide if and only if their corresponding planes E, E' have the same σ-invariant.*

Theorem (explicit computation of $\sigma(E)$ for $n = 4$)

For a plane $E = aW \neq W$ of $\mathrm{PG}(\mathbb{F}_{16}/\mathbb{F}_2)$ we have

$$\sigma(E) = a + a^2 + a^3 + a^4.$$

A further analysis shows that $E \mapsto \sigma(E)$ takes the value $\mathbb{F}_2 = \mathbb{F}_2 1$ precisely 4 times (on the planes of the form $a^3 W$) and is one-to-one on the complementary set of 10 planes.

The Case $v > 7$

or $n = \dim(S) = v - 3 > 4$

Parallels

The number of new planes meeting S in P_1 that can be added to the expurgated code (independence number of the collision graph) still equals the number of values taken by the σ-invariant.

Changes

- Dependence on the plane orbit of W in $\mathrm{PG}(\mathbb{F}_{2^n}/\mathbb{F}_2) \cong \mathrm{PG}(n-1, \mathbb{F}_2)$ (under the Singer+Frobenius action)
  $\Longrightarrow$ Exponential growth
- No explicit formula for the σ-invariant
- There are $2^{n-3} - 1$ cosets $\mathcal{T} + cx^2 + c^2x$, $c \in \mathbb{F}_{2^n} \setminus W$ suitable for removal. Any combination has to be considered.
  $\Longrightarrow$ Doubly exponential growth

For a plane orbit $[W]$ let $T_1, \dots, T_m$ ($m = 2^{n-3} - 1$) be the solids in $\mathrm{PG}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ above W and $\mathbb{F}_{2^n}^\times = \{y_1, \dots, y_{2^n-1}\}$. Define an integral $m \times (2^n - 1)$ matrix $\mathbf{M}_W = (m_{ij})$ by

$$m_{ij} = \#\{E \in T_i; E \neq W \wedge \sigma(E) = y_j\}.$$

Combinatorial optimization problem

Determine the max. local net gain

$$N_1 = \max_{[W]} \max_{\mathbf{x} \in \{0,1\}^m} \bigl(\mathrm{w}_{\mathrm{Ham}}(\mathbf{x}\mathbf{M}_W) - 8\mathrm{w}_{\mathrm{Ham}}(\mathbf{x})\bigr).$$

Example ($v = 8$)

In this case $n = 5$ and the $\begin{bmatrix}5\\3\end{bmatrix}_2 = 155$ planes in $\mathrm{PG}(\mathbb{F}_{32}/\mathbb{F}_2)$ form a single Singer+Frobenius orbit.
Representing $\mathbb{F}_{32}$ as $\mathbb{F}_2[\alpha]$ with $\alpha^5 + \alpha^2 + 1 = 0$, we get

$$\mathbf{M} = \left(\begin{array}{ccc|ccccccccc|cccc} 2 & 2 & 2 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & 0 & 0 & 0 \end{array}\right) \in \mathbb{Z}^{3\times 31}$$

with $\alpha^{23}, \alpha^{25}, \alpha^{28}$ as the first 3 column labels. $\Longrightarrow N_1 = 3$

Experimental Study

using SAGE (`www.sagemath.org`)

v	7	8	9	10	11
n	4	5	6	7	8
N_1	3	3	12	20	≥ 44
$(N_1)_{\text{LMRD}}$	2.33	5	10.33	21	42.33

N_1 Local max. net gain of the EA method

$(N_1)_{\text{LMRD}}$ Local net gain required to equalize the LMRD code bound

Notes

- Algorithm used: Essentially exhaustive search through all Singer+Frobenius orbits and coset combinations. (For $n = 8$ there are 53 orbits and $2^{2^5-1} - 1 = 2^{31} - 1$ coset combinations.
- $\mathcal{C}$ can be further extended by planes meeting S in a line, but computing maximal such extensions is not feasible.

Outline

MOORE's Identity . . .

2-Analog of the Vandermonde determinant evaluation

$$\delta(X_1,\dots,X_k)=\begin{vmatrix} X_1 & X_2 & \dots & X_k \\ X_1^2 & X_2^2 & \dots & X_k^2 \\ X_1^{2^2} & X_2^{2^2} & \dots & X_k^{2^2} \\ \vdots & \vdots & & \vdots \\ X_1^{2^{k-1}} & X_2^{2^{k-1}} & \dots & X_k^{2^{k-1}} \end{vmatrix}$$

$$=\prod_{\lambda\in\mathbb{F}_2^k\setminus\{\mathbf{0}\}}(\lambda_1X_1+\dots+\lambda_kX_k)\quad\text{in } \mathbb{F}_2[X_1,\dots,X_k].$$

Moore's Identity can be proved by induction on k, using

$$\delta(X_1,\dots,X_k)=\delta(X_1,\dots,X_{k-1})\prod_{\lambda\in\mathbb{F}_2^{k-1}}(X_k+\lambda_1X_1+\dots+\lambda_{k-1}X_{k-1}) \tag{1}$$

(in virtually the same way as Vandermonde's Identity).

… Leading to Subspace Polynomials

Suppose U is a k-dimensional $\mathbb{F}_2$-subspace of $\mathbb{F}_2^n$ with basis $\beta_1, \dots, \beta_k$

$$\implies \prod_{u \in U}(X+u) = \prod_{\lambda \in \mathbb{F}_2^k}(X + \lambda_1\beta_1 + \dots + \lambda_k\beta_k)$$
$$= \frac{\delta(\beta_1, \dots, \beta_k, X)}{\delta(\beta_1, \dots, \beta_k)} = \sum_{i=0}^{k} a_i X^{2^i} \in \mathbb{F}_{2^n}[X].$$

Definition

The *subspace polynomial of U* is defined as $\mathrm{s}_U(X) = \prod_{u \in U}(X+u)$.

Properties

- By unique factorization, U is determined by $\mathrm{s}_U(X)$.
- $\mathrm{s}_U(X)$ is a monic, separable (i.e., $a_0 \neq 0$), linearized polynomial in $\mathbb{F}_{2^n}[X]$ of symbolic degree $k = \dim U$.
- Conversely, a polynomial with these properties is a subspace polynomial of $U \subseteq \mathbb{F}_{2^n}$ iff it splits into linear factors in $\mathbb{F}_{2^n}[X]$.

DICKSON Invariants

Definition (from Modular Invariant Theory)

The coefficients of the generic subspace polynomial $\prod(X+\lambda_1 X_1+\cdots+\lambda_k X_k)$ are called *Dickson invariants* and denoted by $\delta_i^{(k)}(X_1,\dots,X_k)$, $1\leq i\leq k$. The indexing is mutatis mutandis the same as for the elementary symmetric polynomials.

Theorem (Dickson)

The ring of $\mathrm{GL}(k,\mathbb{F}_2)$*-invariants in* $\mathbb{F}_2[X_1,\dots,X_k]$ *is freely generated by* δ_i^k*,* $1\leq i\leq k$*.*

Important Consequence

The "Dickson invariant" $\delta_i(U)=\delta_i^{(k)}(\beta_1,\cdots\beta_k)$ is well-defined, and

$$\mathrm{s}_U(X)=X^{2^k}+\delta_1(U)X^{2^k-1}+\cdots+\delta_{k-1}(U)X^2+\delta_k(U)X.$$

For our purposes the most important of these invariants is the *last Dickson invariant*

$$\delta(U)=\delta_k(U)=\prod_{u\in U}u.$$

Examples

Point Polynomials

$\mathrm{s}_P(X) = X(X+a) = X^2 + aX$
for any point $P = \mathbb{F}_2 a$ in $\mathrm{PG}(\mathbb{F}_{2^n}/\mathbb{F}_2) \cong \mathrm{PG}(n-1, \mathbb{F}_2)$

Line Polynomials

For lines $L = \langle a, b \rangle = \{a, b, a+b\}$ in $\mathrm{PG}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ we have

$$\begin{aligned} \mathrm{s}_L(X) &= \left(X^2 + (b^2+ab)X\right) \circ (X^2 + aX) \\ &= (X^2+aX)^2 + (b^2+ab)(X^2+aX) \\ &= X^4 + (a^2+ab+b^2)X^2 + (ab^2+a^2b)X \end{aligned}$$

$\Longrightarrow \delta_1(L) = a^2 + ab + b^2$, $\delta_2(L) = \delta(L) = ab^2 + a^2 b = ab(a+b)$.

Examples (cont'd)

Subspace polynomials in $\mathrm{PG}(\mathbb{F}_{16}/\mathbb{F}_2) \cong \mathrm{PG}(3, \mathbb{F}_2)$

Plane polynomials:

$$
\begin{array}{ll}
W_0 = \{x \in \mathbb{F}_{16}; \mathrm{Tr}(x) = 0\}: & \mathrm{s}_{W_0}(X) = X^8 + X^4 + X^2 + X \\
W = rW_0,\ r \in \mathbb{F}_{16}^\times: & \mathrm{s}_W(X) = X^8 + r^4 X^4 + r^6 X^2 + r^7 X
\end{array}
$$

Line polynomials:

Write $\mathbb{F}_{16}^\times = \langle \xi \rangle$, $\mathbb{F}_4^\times = \langle \omega \rangle$ with $\omega = \xi^5$.
There are 2 Singer+Frobenius line orbits, $[\mathbb{F}_4]$ and $[L_0]$, $L_{)} = \xi^{10}\langle 1, \xi \rangle$, with sizes 5, 30 and line polynnomials

$$\mathrm{s}_{\mathbb{F}_4}[X] = X^4 + X. \quad \mathrm{s}_{L_0}(X) = X^4 + X^2 + \omega X,$$

respectively. The remaining line polynomials are determined from $\mathrm{s}_{rL}(X) = X^4 + r^2 a_1 X^2 + r^3 a_0 X$, $\mathrm{s}_{L^2}(X) = X^4 + a_1^2 X^2 + a_0^2 X$.

Ore's Work

On a Special Class of Polynomials, TAMS 35(1933)

The ring of 2-polynomials

With respect to composition $a(X) \circ b(X) = a\big(b(X)\big)$ ("symbolic multiplication"), the 2-polynomials in $\mathbb{F}_{2^n}[X]$ form a ring L_n. Via $X^{2^i} \mapsto Y^i$, the ring L_n is isomorphic to the skew polynomial ring $\mathbb{F}_{2^n}[Y;\phi]$ with $\phi(a) = a^2$.

The linear map view of 2-polynomials

$\mathrm{End}(\mathbb{F}_{2^n}/\mathbb{F}_2) \cong L_n/(X^{2^n} + X) \cong \mathbb{F}_2[Y;\phi]/(Y^n + 1)$.

Three subspaces associated with U

- $U^\perp$ The orthogonal subspace of U with respect to the trace bilinear form $(x, y) \mapsto \mathrm{Tr}(xy)$.
- U° The *opposite subspace* of U, defined by $\mathrm{s}_U(X) \circ \mathrm{s}_{U^\circ}(X) = \mathrm{s}_{U^\circ}(X) \circ \mathrm{s}_U(X) = X^{2^n} + X$.
- U^* The *adjoint subspace* of U, which may be defined as the subspace $\langle \delta(V)/\delta(U); V \subseteq U \text{ a hyperplane}\rangle$.

Key Facts

Implicit in Ore's work

Relation between $U^\perp$, U°, U^*

$(U^*)^2 = (U^\circ)^\perp$

Theorem

Let U be a k-subspace of $\mathbb{F}_{2^n}$.

1. *$V \mapsto \delta(V)$ maps the $(k+1)$-subspaces of $\mathbb{F}_{2^n}$ containing U bijectively onto the 1-subspaces of the space $\delta(U)U^\circ$. The induced map from $\mathrm{PG}(\mathbb{F}_{2^n})/U$ to $\mathrm{PG}\big(\delta(U)U^\circ\big)$ is a collineation.*
2. *$V \mapsto \delta(V)$ maps the $(k-1)$-subspaces of $\mathbb{F}_{2^n}$ contained in U bijectively onto the 1-subspaces of $\delta(U)U^*$. The induced map from $\mathrm{PG}(U)$ to $\mathrm{PG}\big(\delta(U)U^*\big)$ is a correlation.*

Sketch of proof.

For Part (1) use $\delta(V) = \mathrm{s}_U(x)\delta(U)$ for any β satisfying $V = U + \mathbb{F}_2 x$, together with $U^\circ = \mathrm{Im}\big(x \mapsto \mathrm{s}_U(x)\big)$. For Part (2) the roles of U, V are reversed. □

A Nice Application to Subspace Codes

Corollary

The k-subspaces $U \subseteq \mathbb{F}_{2^v}$ with fixed last Dickson invariant $\delta(U) = a$, $a \in \mathbb{F}_{2^v}^\times$, form a subspace code $\mathcal{C}(a)$ with minimum distance at least 4.

Notes

- By the corollary, the set of k-subspaces of $\mathbb{F}_2^v$ is partitioned into $2^v - 1$ (possibly empty) subspace codes of minimum distance ≥ 4. Viewed as single codes, these are not very interesting, since they are too small. In the case $k = 3$ the largest of these codes has guaranteed size

$$\#\mathcal{C}(a) \geq \frac{1}{2^v - 1}\begin{bmatrix} v \\ 3 \end{bmatrix}_2 = \frac{(2^{v-1} - 1)(2^{v-2} - 1)}{21} \approx \frac{8}{21} \times \#\mathcal{G}.$$

- Compare the corollary with the Gap Theorem in Ben-Sasson et al. 2014.

Outline

The Collision Space

Theorem

The set of multiple values of $\sigma_W(E) = \delta(E)/\delta(Z)^3$, $Z = E \cap W$, is precisely the $(n-3)$-dimensional subspace $(W^2)^\perp$.

Sketch of proof.

For each of the 7 lines (2-dimensional subspaces) $Z \subset W$, $E \mapsto \sigma_W(E)$ maps the planes $E \supset Z$ bijectively to the points in $\delta(Z)^{-2}Z^\circ$, a space of dimension $n-2$. Using $(Z^*)^2 = (Z^\circ)^\perp$, one can show that $\delta(Z)^{-2}Z^\circ = (Z^2)^\perp$.

$$\Longrightarrow (W^2)^\perp = \bigcap_{Z \subset W} \delta(Z)^{-2}Z^\circ.$$

$\Longrightarrow$ The points in $(W^2)^\perp$ (outside $(W^2)^\perp$) have multiplicity 7 (resp., 1), except for the 7 *missing values* $\delta(W)/\delta(Z)^3$. $\square$

Definition

The space $C_W = (W^2)^\perp \subset \mathbb{F}_{2^n}$ is called *collision space* and the corresponding $m \times m$ submatrix $\mathbf{C}_W$ of $\mathbf{M}_W$ *collision matrix* (relative to W).

Simplified optimization problem

Determine the max. local net gain N_1 as the optimal solution of

$$\begin{array}{ll} \text{Maximize} & \sum_{i=1}^{m}(6-r_i)x_i + \mathrm{w}_{\mathrm{Ham}}(\mathbf{x}\mathbf{C}_W) \\ \text{subject to} & \mathbf{x} \in \{0,1\}^m, \end{array} \tag{2}$$

where $r_1, \dots, r_m$ denote the row sums of $\mathbf{C}_W$.

Example ($v = 9$)

There are 7 plane orbits $[W]$ in $\mathrm{PG}(\mathbb{F}_{64}/\mathbb{F}_2)$ with collision matrices

$$\begin{pmatrix} 1&1&2&1&2&0&1\\ 1&1&0&1&0&0&1\\ 1&1&0&1&0&0&1\\ 1&1&0&1&0&0&1\\ 1&1&0&1&0&0&1\\ 1&1&2&1&2&0&1\\ 1&1&2&1&2&4&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&1&1&2\\ 2&2&1&1&1&1&0\\ 0&0&1&1&1&1&0\\ 2&0&1&1&1&1&0\\ 0&2&1&1&1&1&0\\ 2&0&1&1&1&1&2\\ 0&2&1&1&1&1&2 \end{pmatrix}, \begin{pmatrix} 2&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 2&1&1&1&1&1&1\\ 2&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&2&1&1&0&1&1\\ 1&0&1&1&2&1&1\\ 1&0&1&1&0&1&1\\ 1&0&1&1&2&1&1\\ 1&0&1&1&0&1&1\\ 1&2&1&1&2&1&1\\ 1&2&1&1&0&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 2&0&0&1&2&1&0\\ 2&0&2&1&0&1&2\\ 2&2&0&1&0&1&0\\ 0&0&0&1&0&1&2\\ 0&2&2&1&0&1&0\\ 0&2&0&1&2&1&2\\ 0&0&2&1&2&1&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&1&0&0\\ 1&1&2&2&1&0&2\\ 1&1&0&0&1&0&0\\ 1&1&2&2&1&0&2\\ 1&1&0&0&1&0&0\\ 1&1&2&2&1&0&2\\ 1&1&0&0&1&4&0 \end{pmatrix}, \begin{pmatrix} 0&2&0&2&0&2&0\\ 2&0&0&2&2&0&0\\ 0&0&2&2&0&0&2\\ 2&0&0&0&0&2&2\\ 0&0&2&0&2&2&0\\ 2&2&2&0&0&0&0\\ 0&2&0&0&2&0&2 \end{pmatrix}. \implies N_1 = 12$$

Outline

Properties of Collision Matrices

1. Only columns of type 1^7, 2^3 or 4^1 can occur. More precisely, a column labeled with $y \in (W^2)^\perp$ has type 1^7 if y is not a missing value of σ_W (i.e., $y \neq \delta(W)/\delta(Z)^3$ for all lines $Z \subset W$), type 2^3 if y is a missing value of multiplicity 1 (i.e., $y = \delta(W)/\delta(Z)^3$ for exactly one line $Z \subset W$), and type 4^1 if y is a missing value of multiplicity 3 (i.e., $y = \delta(W)/\delta(Z)^3$ for three lines $Z \subset W$). Moreover, Type 4^1 does not occur if n is odd, and occurs at most once as a column of $\mathbf{C}_W$ if n is even.

   1^7 if y is not a missing value of σ_W,
   2^3 if y is a missing value of multiplicity 1,
   4^1 if y is a missing value of multiplicity 3.

   (The multiplicity is the number of lines $Z \subset W$ with $y = \delta(W)/\delta(Z)^3$.)

2. The support of each column is a subspace of $\mathbb{F}_{2^n}/W$.

3. All row sums have the same parity, equal to the parity of the number of columns of type 1^7.

Properties of Collision Matrices (Cont'd)

4. The row sum spectrum of $\mathbf{C}_W$ can be computed from the geometric configuration formed by the multiset of $\mu \leq 7$ missing points of σ_W contained in $(W^2)^\perp$ (in terms of the weight distribution of the associated binary linear $[\mu, k]$ code).
5. Plane orbits $[W]$ with a column of type 4^1 in $\mathbf{C}_W$ (equivalently, with a missing point in $(W^2)^\perp$ of multiplicity 3) can be characterized algebraically: They occur iff n is even and are represented by $W = \langle 1, a, b \rangle$ with a, b satisfying $b^2 + b = \omega(a^2 + a)$, where ω is a generator of $\mathbb{F}_4 \subseteq \mathbb{F}_{2^n}$. The missing points in this case are 1 (of multiplicity 3) and $(b + \omega a + x)^{-3}$ for $x \in \mathbb{F}_4$ (of multiplicity 1).

Proof of the Main Theorem

Idea.

Choose W as the trace-zero plane of the subfield $\mathbb{F}_{16} \subseteq \mathbb{F}_{2^n}$.
$\implies \mathbf{C}_W$ is of the type discussed in Property 5 above. The missing points are 1 (of multiplicity 3) and the primitive 5th roots of unity in $\mathbb{F}_{16}$.

Case 1: $n \equiv 4 \pmod 8$
In this case $(W^2)^\perp \cap \mathbb{F}_{16} = \mathbb{F}_2$
$\implies$ 1 is the only missing point contained in $(W^2)^\perp$.
$\implies \mathbf{C}_W$ has row sums 4 and 10 with corresponding frequencies $f_4 = 2^{n-4}$, $f_{10} = 2^{n-4} - 1$.
This leads to the stated lower bound for the max. (global) net gain.

Case 2: $n \equiv 0 \pmod 8$
In this case $F_{16} \subset (W^2)^\perp$, so that $(W^2)^\perp$ contains all $3 + 1 + 1 + 1 + 1 = 7$ missing points.
The proof is similar to that in Case 1 but more difficult. One can show that $N_1 \geq 2^{n-8} \times 54$ using $n = 8$ as an “anchor”. □

Outline

Open Problems/Future Work

- We conjecture that the main theorem remains true for all lengths $v \geq 7$, $v \neq 8, 10$. Prove this conjecture!
For the yet unsettled case $v \equiv 1 \pmod 4$, or $n \equiv 2 \pmod 4$, there is overwhelming computational evidence for the truth. (Here it suffices to exhibit a plane $W = \langle 1, a, b \rangle$ of the type considered in Case 1 of the proof.)
- Use Expurgation-Augmentation with non-Gabidulin MRD codes.
- Investigate non-standard rearrangements of free lines into new planes.
- Determine the structure of the set of free planes of $\mathcal{C}$ meeting S in a line, and use this structure to solve the extension problem efficiently.
- Generalize Expurgation-Augmentation to constant dimensions $k > 3$.

Thank You

Outline

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