Large Sets of t-designs with large blocks

Reinhard Laue
Universität Bayreuth,
laue@uni-bayreuth.de

joint work with Michael Kiermaier

April 4, 2016

V set of v elements, $\binom{V}{k}$ set of k-element subsets of V

$\mathcal{D} = (V, \mathcal{B})$ with $\mathcal{B} \subseteq \binom{V}{k}$ is a $t - (v, k, \lambda)$ **design:**

$$\forall T \in \binom{V}{t} \; |\{B \in \mathcal{B} \mid T \subseteq B\}| = \lambda$$

$LS[N](t, k, v)$ **Large set:**

$\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ partition of $\binom{V}{k}$,
each $\mathcal{D}_i = (V, \mathcal{B}_i)$ a $t - (v, k, \lambda)$ design

Theory of Large Sets of t-designs:
Lecture Notes of Khosrovshahi and Tayfeh-Rezaie (2004)

- Any $LS[N](t,k,v)$ is an $LS[N](s,k,v)$, for $1 \le s \le t$
- $LS[N](t,k,v)$ admissible: $N \mid \binom{v-i}{k-i}$ for $0 \le i \le t$
- formally
  $LS[N](-1,k,v)$: Any subset of $\binom{V}{k}$
- $LS[N](0,k,v)$: exists if N divides $\binom{V}{k}$
- $LS[N](1,k,v)$: exists if admissible (Baranyai)
- $LS[5](2,3,7)$, $LS[7](3,4,10)$, $LS[11](4,5,15)$: **Not** realizable
- Teirlinck: For all t there exist $LS[N](t,k,v)$ (parameters astronomical)
- Khosrovshahi and Ajoodani-Namini: Recursive construction of Large Sets with small parameters

Assume an $LS[N](t, k, v)$, then there exist:

- $LS[N](t, v-k, v)$, the Dual
- $LS[N](t-1, k-1, v-1)$, a Derived
- $LS[N](t-1, k, v-1)$, a Residual

- **TvT** (Tran van Trung, van Leijenhorst, Driessen)
  $\exists LS[N](t-1, k-1, v-1)$,
  $\exists LS[N](t-1, k, v-1)$

$$\implies \exists LS[N](t-1, k, v)$$

Constructions

Direct constructions:

Combine orbits of a group on $\binom{V}{k}$ to block sets $\mathcal{B}_i$ forming a Large Set:

Hand computations ($t = 2$)
or
Computer algorithms (DISCRETA)

Recursive constructions:

Apply rules to the directly constructed Large Sets.

Existence of LS[7](3,k,v)

```
realizable t-values                              v

                                          0 0 0  7
                                         - 1 1 1 8
                                        - - 2 2  9
                                       - - - 2 3 10
                                      - - - - 3  11
                                     - - - - - 3 12
                                    - - - - - -  13
                                   0 0 0 0 0 0 - 14
                                  - 1 1 1 1 1 -  15
                                 - - 2 2 2 2 - - 16
                                - - - 3 3 3 - -  17
                               - - - - 3 3 - - - 18
                              - - - - - 3 - - -  19
                             - - - - - - - - - - 20
                            0 0 0 0 0 0 - 0 0 0  21
```

```
                        - 1 1 1 1 1 - - 1 1 1 22
                       - - 2 2 2 2 - - - 2 2  23
                      - - - 3 3 3 - - - - 3 3 24
                     - - - - 3 3 - - - - - 3  25
                    - - - - - 3 - - - - - - 3 26
                   - - - - - - - - - - - - -  27
                  0 0 0 0 0 0 - 0 0 0 0 0 0 - 28
                 - 1 1 1 1 1 - - 1 1 1 1 1 -  29
                - - 2 2 2 2 - - - 2 2 2 2 - - 30
               - - - 3 3 3 - - - - 3 3 3 - -  31
              - - - - 4 3 - - - - - 3 3 - - - 32
             - - - - - 3 - - - - - - 3 - - -  33
            - - - - - - - - - - - - - - - - - 34
           0 0 0 0 0 0 - 0 0 0 0 0 0 - 0 0 0  35
          - 1 1 1 1 1 - - 1 1 1 1 1 - - 1 1 1 36
         - - 2 2 2 2 - - - 2 2 2 2 - - - 2 2  37
        - - - 3 3 3 - - - - 3 3 3 - - - - 3 3 38
       - - - - 3 3 - - - - - 3 3 - - - - - 3  39
```

```
         - - - - - 3 - - - - - - 3 - - - - - - 3 40
        - - - - - - - - - - - - - - - - - - - -  41
       0 0 0 0 0 0 - 0 0 0 0 0 0 - 0 0 0 0 0 0 - 42
      - 1 1 1 1 1 - - 1 1 1 1 1 - - 1 1 1 1 1 -  43
     - - 2 2 2 2 - - - 2 2 2 2 - - - 2 2 2 2 - - 44
    - - - 3 3 3 - - - - 3 3 3 - - - - 3 3 3 - -  45
   - - - - 3 3 - - - - - 3 3 - - - - - 3 3 - - - 46
  - - - - - 3 - - - - - - 3 - - - - - - 3 - - -  47
 - - - - - - - - - - - - - - - - - - - - - - - - 48
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0  49
 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 50
- 2 2 2 2 1 1 1 2 2 2 2 1 1 1 2 2 2 2 1 1 1 2 2  51
 - 3 3 3 1 1 1 1 3 3 3 1 1 1 1 3 3 3 1 1 1 1 3 3 52
```

Existence of LS[2](7,k,v)

```
realizable t-values                             v
                                                1
                                              0 2
                                             -  3
                                            0 0 4
                                           - 1  5
                                          0 - 2 6
                                         - - -  7
                                        0 0 0 0 8
                                       - 1 1 1  9
                                      0 - 2 2 2 10
                                     - - - 3 3  11
                                    0 0 0 - 4 4 12
                                   - 1 1 - - 5  13
                                  0 - 2 - 0 - 6 14
                                 - - - - - - -  15
                                0 0 0 0 0 0 0 0 16
                               - 1 1 1 1 1 1 1  17
                              0 - 2 2 2 2 2 2 2 18
                             - - - 3 3 3 3 3 3  19
                            0 0 0 - 4 4 4 4 4 4 20
                           - 1 1 - - 5 5 5 5 5  21
                          0 - 2 - 0 - 6 6 6 5 6 22
                         - - - - - - - 6 6 6 5  23
t = 7:                  0 0 0 0 0 0 0 - 6 7 6 6 24
                       - 1 1 1 1 1 1 - - 6 6 6  25
t = 7:                0 - 2 2 2 2 2 - 0 - 6 6 7 26
                     - - - 3 3 3 3 - - - - 6 6  27
t = 7:              0 0 0 - 4 4 4 - 0 0 0 - 6 7 28
                   - 1 1 - - 5 5 - - 1 1 - - 6  29
t = 7:            0 - 2 - 0 - 6 - 0 - 2 - 0 - 7 30
                 - - - - - - - - - - - - - - -  31
                0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 32
               - 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1  33
              0 - 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 34
```

```
                                 - - - 3 3 3 3 3 3 3 3 3 3 3 3 2 2  35
                                0 0 0 - 4 4 4 4 4 4 4 4 4 4 4 2 2 3 36
                               - 1 1 - - 5 5 5 5 5 5 5 5 5 5 2 2 2  37
                              0 - 2 - 0 - 6 6 6 6 6 5 6 6 6 3 3 3 3 38
                             - - - - - - - 6 6 6 6 6 6 6 6 3 3 3 3  39
                            0 0 0 0 0 0 0 - 6 7 6 6 6 7 6 4 4 4 4 4 40
                           - 1 1 1 1 1 1 - - 6 6 6 6 6 6 4 4 4 4 4  41
                          0 - 2 2 2 2 2 - 0 - 6 6 7 6 6 5 5 5 5 5 5 42
                         - - - 3 3 3 3 - - - - 6 6 6 6 5 5 5 5 5 5  43
                        0 0 0 - 4 4 4 - 0 0 0 - 6 7 6 6 6 6 6 5 6 6 44
                       - 1 1 - - 5 5 - - 1 1 - - 6 6 6 6 6 6 5 5 6  45
                      0 - 2 - 0 - 6 - 0 - 2 - 0 - 7 6 7 6 7 6 6 5 7 46
                     - - - - - - - - - - - - - - - 6 6 6 6 6 6 5 5  47
                    0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 - 7 6 7 7 7 6 6 6 48
                   - 1 1 1 1 1 1 1 1 1 1 1 1 1 1 - - 6 6 7 7 6 6 6  49
                  0 - 2 2 2 2 2 2 2 2 2 2 2 2 2 - 0 - 7 6 8 6 7 6 7 50
                 - - - 3 3 3 3 3 3 3 3 3 3 3 3 - - - - 6 6 6 6 6 6  51
                0 0 0 - 4 4 4 4 4 4 4 4 4 4 4 - 0 0 0 - 7 6 7 6 7 7 52
               - 1 1 - - 5 5 5 5 5 5 5 5 5 5 - - 1 1 - - 6 6 6 6 7  53
              0 - 2 - 0 - 6 6 6 6 6 6 6 6 6 - 0 - 2 - 0 - 7 6 7 6 8 54
             - - - - - - - 6 6 6 6 6 6 6 6 - - - - - - - - 6 6 6 6  55
            0 0 0 0 0 0 0 - 6 7 6 6 6 7 6 - 0 0 0 0 0 0 0 - 7 7 7 7 56
           - 1 1 1 1 1 1 - - 6 6 6 6 6 6 - - 1 1 1 1 1 1 - - 7 7 7  57
          0 - 2 2 2 2 2 - 0 - 6 6 7 6 6 - 0 - 2 2 2 2 2 - 0 - 7 7 8 58
         - - - 3 3 3 3 - - - - 6 6 6 6 - - - - 3 3 3 3 - - - - 7 7  59
        0 0 0 - 4 4 4 - 0 0 0 - 6 7 6 - 0 0 0 - 4 4 4 - 0 0 0 - 7 8 60
       - 1 1 - - 5 5 - - 1 1 - - 6 6 - - 1 1 - - 5 5 - - 1 1 - - 7  61
      0 - 2 - 0 - 6 - 0 - 2 - 0 - 7 - 0 - 2 - 0 - 6 - 0 - 2 - 0 - 8 62
     - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -  63
    0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 64
   - 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1  65
  0 - 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 66
 - - - 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 2 2  67
0 0 0 - 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 4 4 4 4 4 4 4 4 4 4 4 2 2 3 68
```

$|\mathbf{V}| = \mathbf{v} = \mathbf{k} + \mathbf{n}$

Grid graph for $\binom{v}{k}$, $0 \leq k \leq v$:

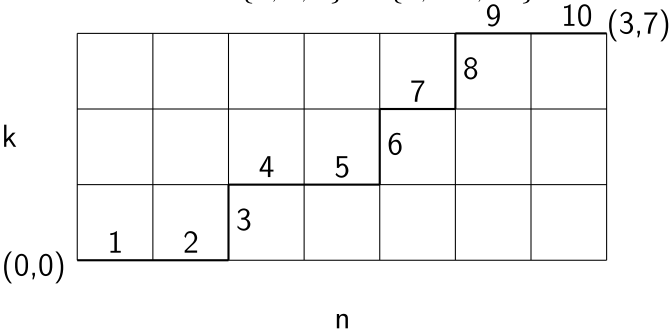
Vertex set: $\{(i,j) \mid 0 \leq i \leq k,\ 0 \leq j \leq n\}$,

Egde set: $\{(i,j) \to (i+1,j) \mid 0 \leq i < k,\ 0 \leq j \leq n\}$

$\cup \{(i,j) \to (i,j+1) \mid 0 \leq i \leq k,\ 0 \leq j < n\}$

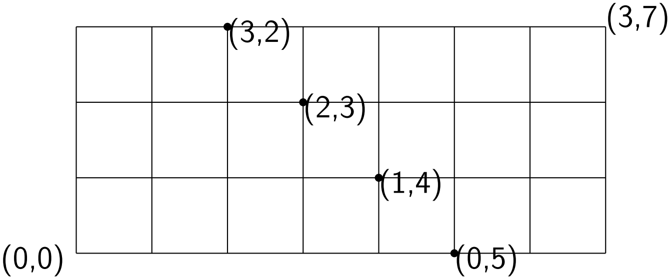
Path: $((0,0) \to \cdots \to (k,n)) \longleftrightarrow K \in \binom{V}{k}$.

A Grid: Path for $\{3,6,8\} \subset \{1,\ldots,10\}$

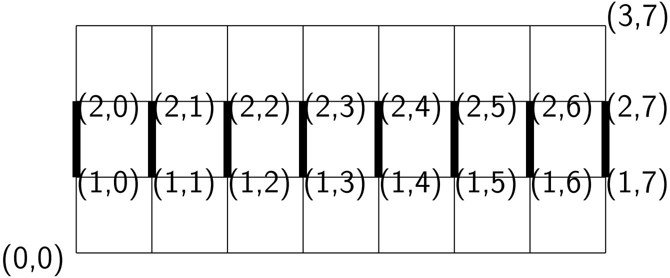


Cut set describes partition of $\binom{v}{k}$:

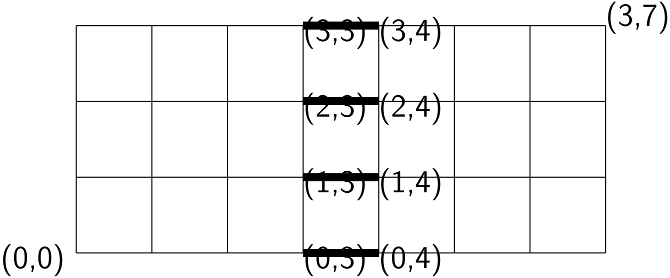
Cut Vertices



Covering Cut Edges



Avoiding Cut Edges



x cut element:

Join Large Set on $[(0,0),x]$ **with Large Set on** $[x,(k,n)]$**:**

$\mathcal{K}_1$, set of paths to x,

running through x, combine with

$\mathcal{K}_2$, set of paths from x.

3 types of joins: $\mathcal{B}_1 * \mathcal{B}_2 \subseteq \binom{V}{k}$

cut	x	*join* $*$	k	$v = \vert V \vert$
point	$\cdot$	*ordinary*	$k_1 + k_2$	$v_1 + v_2$
edge	$-$	*avoiding*	$k_1 + k_2$	$v_1 + v_2 + 1$
edge	$\vert$	*covering*	$k_1 + k_2 + 1$	$v_1 + v_2 + 1$

Basic Theorem (Ajoodani-Namini, Khosrovshahi)

$$\binom{V}{k} = \mathcal{B}_1^{(1)} * \mathcal{B}_2^{(1)} \dot\cup \ldots \dot\cup\, \mathcal{B}_1^{(m)} * \mathcal{B}_2^{(m)}$$

$$\forall \mathcal{B}_j^{(i)}\ \exists LS[N](t_j^{(i)}, k_j^{(i)}, v_j^{(i)})$$

$$\forall i\ :\ t_1^{(i)} + t_2^{(i)} + 1 \geq t$$

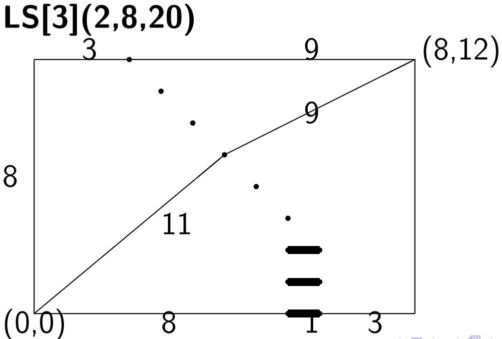
$$\Longrightarrow \exists LS[N](t, k, v),$$

Use of Avoiding Joins mixed with Ordinary Joins

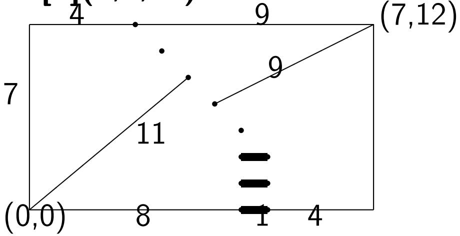
Theorem (Khosrovshahi, Ajoodani-Namini):
If there exist $LS[N](t,k,v)$ for $k = t+1, \dots m$ then there exist $LS[N](t,k,v+l(v-t))$ for $k = t+1, \dots m$ and all positive integers l.

Construction of $LS[3](2, k, 20)$ for $3 \leq k \leq 8$:

Start from $LS[3](2, k, 11)$ for $3 \leq k \leq 8$

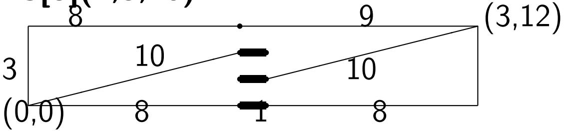


LS[3](2,7,20)



. . .

LS[3](2,3,20)



The last 4 rows use Derived Large sets in each case.

Adding 9 columns on the left side to the grid and using the new results yields $LS[3](2, k, 29)$ for $3 \leq k \leq 8$, induction gives infinite series.

Use of Covering Joins

Theorem Let $v = k_1 + 1 + v_2 = k_2 + 1 + v_1$ and $m = v_1 - k_1 = v_2 - k_2$.
For $0 \le i \le m$
$\exists LS[N](t_1^{(i)}, k_1, k_1 + i) \wedge \exists LS[N](t_2^{(i)}, k_2, v_2^{(i)} - i)$
such that $\forall i \quad t_1^{(i)} + t_2^{(i)} + 1 \ge t$

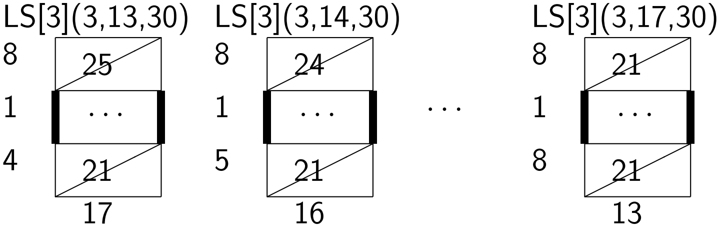
$$\implies \exists LS[N](t, k, v).$$

By induction:
Theorem $LS[N](t, k, v)$ for all $k \in (t, v - t)$ yield $LS[N](t, \hat{k}, v + l(v - t))$ for all positive integers l and all $\hat{k} \bmod (v - t) \in (t, v - t)$.

Construction of $LS[3](3, 13, 30)$:

$$
\begin{array}{lcll}
\mathcal{B}^{(0)} & : & LS[3](-1,4,4) & *LS[3](3,8,25) \\
\mathcal{B}^{(1)} & : & LS[3](-1,4,5) & *LS[3](3,8,24) \\
\mathcal{B}^{(2)} & : & LS[3](-1,4,6) & *LS[3](3,8,23) \\
\mathcal{B}^{(3)} & : & LS[3](-1,4,7) & *LS[3](3,8,22) \\
\mathcal{B}^{(4)} & : & LS[3](-1,4,8) & *LS[3](3,8,21) \\
\mathcal{B}^{(5)} & : & LS[3](0,4,9) & *LS[3](2,8,20) \\
\mathcal{B}^{(6)} & : & LS[3](1,4,10) & *LS[3](1,8,19) \\
\mathcal{B}^{(7)} & : & LS[3](2,4,11) & *LS[3](0,8,18) \\
\mathcal{B}^{(8)} & : & LS[3](3,4,12) & *LS[3](-1,8,17) \\
\mathcal{B}^{(9)} & : & LS[3](-1,4,13) & *LS[3](3,8,16) \\
\mathcal{B}^{(10)} & : & LS[3](-1,4,14) & *LS[3](3,8,15) \\
\mathcal{B}^{(11)} & : & LS[3](-1,4,15) & *LS[3](3,8,14) \\
\mathcal{B}^{(12)} & : & LS[3](-1,4,16) & *LS[3](3,8,13) \\
\mathcal{B}^{(13)} & : & LS[3](-1,4,17) & *LS[3](3,8,12) \\
\mathcal{B}^{(14)} & : & LS[3](0,4,18) & *LS[3](2,8,11) \\
\mathcal{B}^{(15)} & : & LS[3](1,4,19) & *LS[3](1,8,10) \\
\mathcal{B}^{(16)} & : & LS[3](2,4,20) & *LS[3](0,8,9) \\
\mathcal{B}^{(17)} & : & LS[3](3,4,21) & *LS[3](-1,8,8)
\end{array}
$$

Use $LS[3](3, k, 21)$ for $4 \leq k \leq 8$ to construct $LS[3](3, k, 30)$ for $13 \leq k \leq 17$



Adding 9 new rows below and using the new results yields $LS[3](3, k, 39)$ for $k = 22, \ldots, 26$.

Adding 9 new columns to the left keeps the values of k but increases v to $v + 9$.

Both actions can be iterated recursively.

Starters:

Table: $LS[5](t, k, v)$ from prescribed groups of automorphisms

Group	Large Set
C_{13}	$LS[5](3, 4, 13)$
$C_{17}+$	$LS[5](3, 4, 18)$
$Hol(C_{17})^4+$	$LS[5](3, 9, 18)$
$PSL(2, 23)-$	$LS[5](3, 9, 23)$
$PSL(2, 27)-$	$LS[5](2, k, 27)\ \ k = 7, 12$
$D_7 \times D_4$	$LS[5](3, 6, 28)$
$PSL(2, 27)$	$LS[5](3, 8, 28)$
$PSL(2, 31)$	$LS[5](3, 11, 32)$
$PGL(2, 32)$	$LS[5](4, k, 33),\ \ k = 9, 10$

$Hol(C_{47})^2$	$LS[5](2,k,47),\ \ k=3,4$
$Hol(C_{67})^2$	$LS[5](2,k,67)\ \ k=8,9$
$PSL(2,79)$	$LS[5](4,6,80)$
$PSL(2,127)-$	$LS[5](2,k,127), k=25,26,27,50,51,52$
$PSL(2,127)$	$LS[5](3,6,128)$

Recursion:
LS[5](2,k,v) admissible, $v \neq 7$:

$$k \bmod 125 \in \{3,\ldots,124\} \Longrightarrow \exists LS[5](2,k,v)$$

Previously $k \leq 8$.

Recursion for selected Parameters

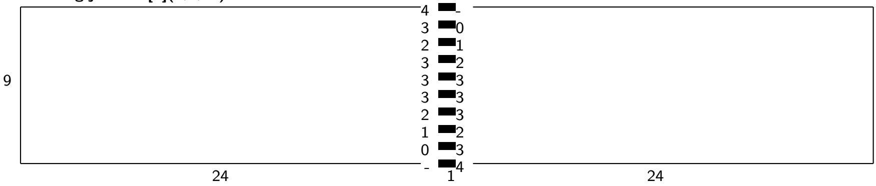
```
N = 5                                         1
                                             - 2
                                            -  3
                                           - - 4
                                          0 0  5
                                         - 1 1 6
                                        - - 1  7
                                       - - - 2 8
                                      - - - -  9
                                     0 0 0 0 - 10
                                    - 1 1 1 -  11
                                   - - 2 2 - - 12
                                  - - - 3 - -  13
                                 - - - - - - - 14
                                0 0 0 0 - 0 0  15
                               - 1 1 1 - - 1 1 16
                              - - 2 2 - - - 2  17
                             - - - 3 - - - - 3 18
                            - - - - - - - - -  19
                           0 0 0 0 - 0 0 0 0 - 20
                          - 1 1 1 - - 1 1 1 -  21
                         - - 2 2 - - - 2 2 - - 22
                        - - - 3 - - - - 3 - -  23
                       - - - - - - - - - - - - 24
                      0 0 0 0 0 0 0 0 0 0 0 0  25
                     - 1 1 1 1 1 1 1 1 1 1 1 1 26
                    - - 2 2 2 2 2 2 2 2 2 2 2  27
                   - - - 3 3 3 2 3 3 2 3 2 2 3 28
                  - - - - 3 3 2 2 3 2 2 2 2 2  29
                 0 0 0 0 - 3 2 2 2 2 3 2 2 2 3 30
                - 1 1 1 - - 2 2 2 2 2 2 2 2 2  31
               - - 2 2 - - - 3 3 3 3 2 3 2 2 3 32
t = 4 -->     - - - 3 - - - - 4 4 3 2 2 3 2 2  33
             - - - - - - - - - 4 3 2 2 2 2 2 3 34
            0 0 0 0 - 0 0 0 0 - 3 2 2 2 2 2 2  35
```

```
                                    - 1 1 1 - - 1 1 1 - - 2 2 2 2 2 2 3 36
                                  - - 2 2 - - - 2 2 - - - 3 3 2 3 2 2  37
                                - - - 3 - - - - 3 - - - - 3 2 2 2 2 3 38
                              - - - - - - - - - - - - - - 2 2 2 2 2  39
                            0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 2 2 2 2 3 40
                          - 1 1 1 - - 1 1 1 - - 1 1 1 - - 2 2 2 2  41
                        - - 2 2 - - - 2 2 - - - 2 2 - - - 2 2 2 3 42
                      - - - 3 - - - - 3 - - - - 3 - - - - 3 2 2  43
                    - - - - - - - - - - - - - - - - - - - 2 2 3 44
                  0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 3 3  45
                - 1 1 1 - - 1 1 1 - - 1 1 1 - - 1 1 1 - - 3 3 46
              - - 2 2 - - - 2 2 - - - 2 2 - - - 2 2 - - - 3  47
            - - - 3 - - - - 3 - - - - 3 - - - - 3 - - - - 3 48
          - - - - - - - - - - - - - - - - - - - - - - - -  49
        0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 - 50
      - 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 -  51
    - - 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 - - 52
  - - - 3 3 3 3 3 3 2 3 2 2 3 2 2 3 2 3 3 2 3 3 3 - -  53
- - - - 3 3 4 3 3 2 2 2 2 2 2 2 2 2 2 3 2 2 3 3 - - - 54
0 0 0 0 - 3 3 3 3 2 3 2 2 2 3 2 2 2 3 2 2 2 2 3 - 0 0  55
- 1 1 1 - - 3 3 3 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 - - 1 1 56
- - 2 2 - - - 3 3 3 3 2 3 3 2 3 2 2 3 2 3 3 3 3 - - - 2  57
t = 4 -->    - - - 3 - - - - 4 4 3 2 2 3 2 2 2 2 3 2 2 3 4 4 - - - - 3 58
- - - - - - - - - - 4 3 2 2 2 2 2 3 2 2 2 2 2 3 4 - - - - -  59
0 0 0 0 - 0 0 0 0 - 3 2 3 3 2 2 3 2 3 2 3 3 3 3 - 0 0 0 0 - 60
- 1 1 1 - - 1 1 1 - - 2 2 3 2 2 2 3 2 2 2 3 3 3 - - 1 1 1 -  61
- - 2 2 - - - 2 2 - - - 3 3 2 3 2 2 3 2 3 2 3 3 - - - 2 2 - - 62
- - - 3 - - - - 3 - - - - 3 2 2 2 2 3 2 2 2 2 3 - - - - 3 - -  63
- - - - - - - - - - - - - - 2 2 2 2 2 2 2 2 2 2 - - - - - - - - 64
0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 3 3 3 3 3 3 3 3 3 - 0 0 0 0 - 0 0  65
- 1 1 1 - - 1 1 1 - - 1 1 1 - - 3 3 3 3 3 3 3 3 - - 1 1 1 - - 1 1 66
- - 2 2 - - - 2 2 - - - 2 2 - - - 3 3 3 3 3 3 3 - - - 2 2 - - - 2  67
- - - 3 - - - - 3 - - - - 3 - - - - 3 3 3 3 3 3 - - - - 3 - - - - 3 68
- - - - - - - - - - - - - - - - - - - 3 3 3 3 3 - - - - - - - - - -  69
0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 4 4 4 4 - 0 0 0 0 - 0 0 0 0 - 70
- 1 1 1 - - 1 1 1 - - 1 1 1 - - 1 1 1 - - 4 4 4 - - 1 1 1 - - 1 1 1 -  71
- - 2 2 - - - 2 2 - - - 2 2 - - - 2 2 - - - 4 4 - - - 2 2 - - - 2 2 - - 72
```

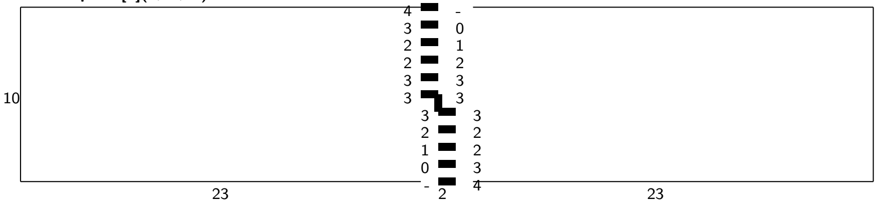
```
- - - 3 - - - - 3 - - - - 3 - - - - 3 - - - - 4 - - - - 3 - - - - 3 - - 73
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - 74
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 - 0 0 0 0 0 0 0 0 0 0 0 0 75
- 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 - - 1 1 1 1 1 1 1 1 1 1 1 1 76
- - 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 - - - 2 2 2 2 2 2 2 2 2 2 2 77
- - - 3 3 3 2 3 3 2 3 2 2 3 2 2 3 2 3 3 2 3 3 3 - - - - 3 3 3 2 3 3 2 3 2 2 3 78
- - - - 3 3 2 2 3 2 2 2 2 2 2 2 2 2 2 3 2 2 3 3 - - - - - 3 3 2 2 3 2 2 2 2 2 79
0 0 0 0 - 4 2 2 2 2 3 2 2 2 3 2 2 2 3 2 2 2 2 3 - 0 0 0 0 - 3 2 2 2 2 3 2 2 2 3 80
- 1 1 1 - - 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 - - 1 1 1 - - 2 2 2 2 2 2 2 2 2 81
- - 2 2 - - - 3 3 3 3 2 3 3 2 3 2 2 3 2 3 3 3 3 - - - 2 2 - - - 3 3 3 3 2 3 3 2 3 82
- - - 3 - - - - 4 4 3 2 2 3 2 2 2 2 3 2 2 3 4 4 - - - - 3 - - - - 4 4 3 2 2 3 2 2 83
- - - - - - - - - 4 3 2 2 2 2 2 3 2 2 2 2 2 3 4 - - - - - - - - - - 4 3 2 2 2 2 2 3 84
0 0 0 0 - 0 0 0 0 - 3 2 2 2 2 2 2 2 2 2 2 2 2 3 - 0 0 0 0 - 0 0 0 0 - 3 2 2 2 2 2 2 85
- 1 1 1 - - 1 1 1 - - 2 2 2 2 2 2 3 2 2 2 2 2 2 - - 1 1 1 - - 1 1 1 - - 2 2 2 2 2 2 3 86
- - 2 2 - - - 2 2 - - - 3 3 2 3 2 2 3 2 3 2 3 3 - - - 2 2 - - - 2 2 - - - 3 3 2 3 2 3 87
- - - 3 - - - - 3 - - - - 3 2 2 2 2 3 2 2 2 2 3 - - - - 3 - - - - 3 - - - - 3 2 2 2 2 3 88
- - - - - - - - - - - - - - 2 2 2 2 2 2 2 2 2 2 - - - - - - - - - - - - - - - 2 2 2 2 2 89
0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 3 3 3 3 3 3 3 3 3 - 0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 3 3 3 3 3 90
- 1 1 1 - - 1 1 1 - - 1 1 1 - - 3 3 3 3 3 3 3 3 - - 1 1 1 - - 1 1 1 - - 1 1 1 - - 3 3 3 3 91
- - 2 2 - - - 2 2 - - - 2 2 - - - 3 3 3 3 3 3 3 - - - 2 2 - - - 2 2 - - - 2 2 - - - 3 3 3 3 92
- - - 3 - - - - 3 - - - - 3 - - - - 3 3 3 3 3 3 - - - - 3 - - - - 3 - - - - 3 - - - - 3 3 3 93
- - - - - - - - - - - - - - - - - - - 3 3 3 3 3 - - - - - - - - - - - - - - - - - - - - 3 3 3 94
0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 4 4 4 4 - 0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 0 0 0 0 - 4 4 95
- 1 1 1 - - 1 1 1 - - 1 1 1 - - 1 1 1 - - 4 4 4 - - 1 1 1 - - 1 1 1 - - 1 1 1 - - 1 1 1 - - 4 5 96
- - 2 2 - - - 2 2 - - - 2 2 - - - 2 2 - - - 4 4 - - - 2 2 - - - 2 2 - - - 2 2 - - - 2 2 - - - 4 97
- - - 3 - - - - 3 - - - - 3 - - - - 3 - - - - 4 - - - - 3 - - - - 3 - - - - 3 - - - - 3 - - - - 5 98
```

$$\exists LS[5](4, k, 33),\ \ k = 9, 10 \Longrightarrow \exists LS[5](4, k, 58),\ \ k = 9, 10$$

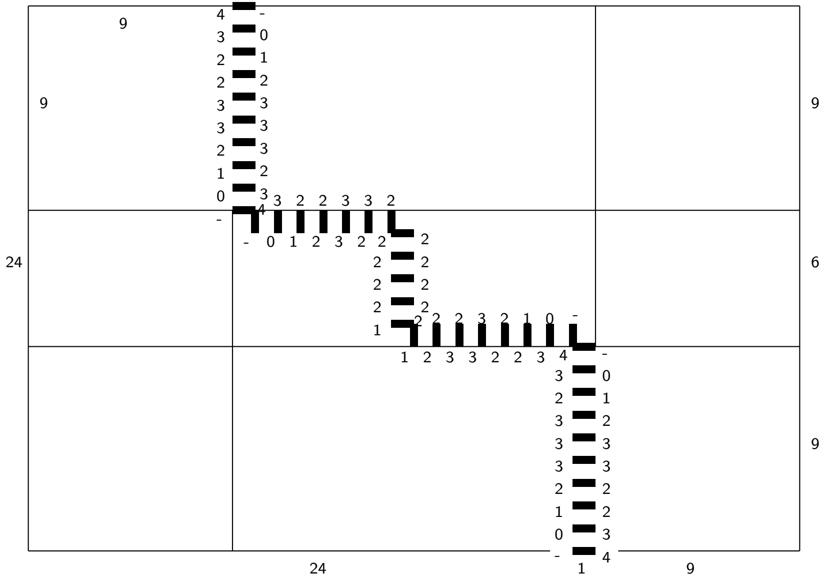
Avoiding join: LS[5](4,9,58)



One Step: LS[5](4,10,58)



Staircase: LS[5](4,24,58)



Recursion:
Enlarge point set by 25 points,
or
Form dual Large Set.
Example:

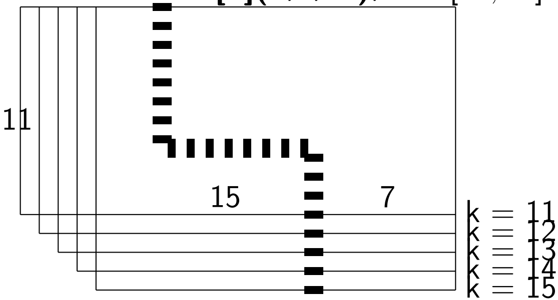
- $LS[5](4, 9, 33 + n25)$,
- $LS[5](4, 24, 33 + n25)$,
- $LS[5](4, 34, 58 + n25)$,
- $LS[5](4, 49, 58 + n25)$,
- $LS[5](4, 59, 83 + n25)$,
- $LS[5](4, 74, 83 + n25)$,
- etc.

$$\Longrightarrow \exists LS[5](4, k, 33 + n25)$$

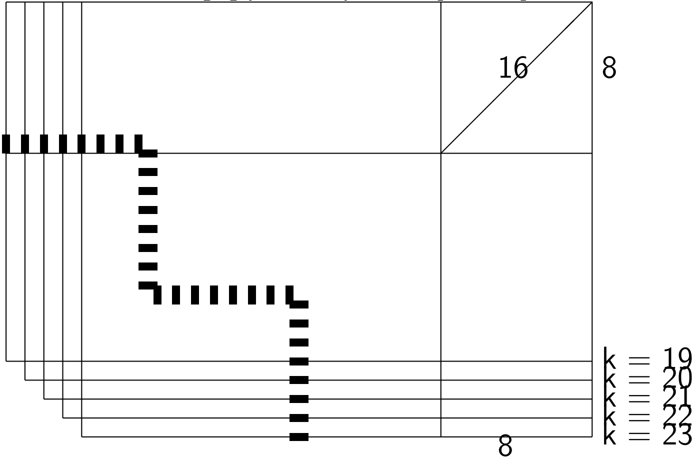
for $k \pmod{25} \in \{9, 10, 23, 24\}$ and $0 < k < v$.
Similarly for $LS[2](7, 14, v)$.

Starters: $LS[4](2,3,10)$, $LS[4](2,k,18)$, $k=3,\dots,7$.

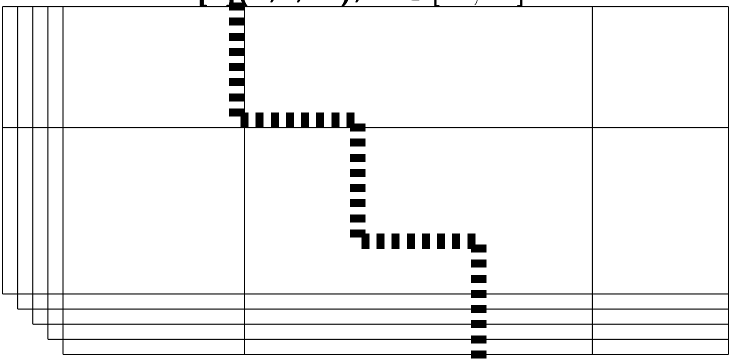
Staircase: LS[4](2,k,34), $k \in [11,15]$



Staircase: LS[4](2,k,50), $k \in [19, 23]$



Staircase: LS[4](2,k,66), $k \in [19, 23]$



$v \in \{34, \ldots, 38\}$, $k \in \{11, \ldots, 15\}, k + 8m < v + 16n$:

$$\Longrightarrow \exists LS[4](2, k + 8m, v + 16n)$$

Table: Existence of large sets

N	t	k	v
2	3	$3 < k < 8$	$v \bmod 8 \equiv 3$
2	4	$k \bmod 32 \notin \{16, \ldots, 19\}$	$v \bmod 32 \equiv 3$
2	5	$k \bmod 64 \notin \{32, \ldots, 35\}$	$v \bmod 64 \equiv 3$
2	6	$k \bmod 16 \notin \{10, 12\}16$	$v \bmod 16 \equiv 6$
2	7	$k \bmod 16 \in \{10, 14\}$	$v \bmod 16 \equiv 6, 10$
3	4	$k \bmod 27 \notin \{11, 12, 13, 18, 19, 20\}$	$v \bmod 27 \equiv 4$
3	6	$k \bmod 33 \equiv 10, 23$	$v \bmod 27 \equiv 6$
4	2	$k \bmod 8 \in \{3, \ldots, 7\}$	$v \bmod 16 \in \{2, \ldots, 6\}$
4	3	$k \bmod 8 \in \{4, \ldots, 7\}$	$v \bmod 16 \equiv 3$
5	2	≤ 124	$\neq 7$
5	3	$k \bmod 5 \equiv 4$	$v \bmod 5 \equiv 3, \neq 8$
5	3	$k \bmod 25 \notin \{7, 10, 12, 13, 14, 15, 17, 20\}$	$v \bmod 25 \equiv 3$

Prime p: $\forall k \in \{t+1, \ldots, p-1\} \exists LS[p](t, k, p+t)$

$$\implies \forall \hat{k} \bmod p \equiv k, v \bmod p \equiv t \exists LS[p](t, \hat{k}, v)$$

Yields:

p	t	v
7	2	$*$
7	3	$\neq 10$
11	2	$*$
13	2	≥ 67
17	2	$*$
29	2	$*$
19	2	≥ 78

Further Large Sets then result from standard methods.

Comment

The shown recursion techniques work as well for vector space designs.
$LS_2[3](2, k, 8), \ k = 3, 4, 5$
$LS_3[2](2, 3, 6), \ LS_5[2](2, 3, 6)$

Find starters!

Thank You