On the density of cyclotomic lattices constructed from codes

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Network Coding and Designs
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supervised by:
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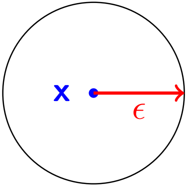
Outline

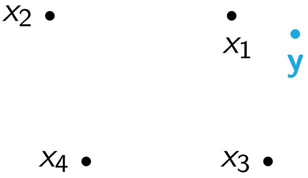
Motivation: Decoding without Ambiguity

Consider a noisy channel over $\mathbb{R}^n$: suppose there exists ϵ such that if $x \in \mathbb{R}^n$ is sent, with high probability, the received vector y is in $B(x, \epsilon)$:



Motivation: Decoding without Ambiguity

If there is only one codeword in the ball of radius ϵ centred in the received vector y,



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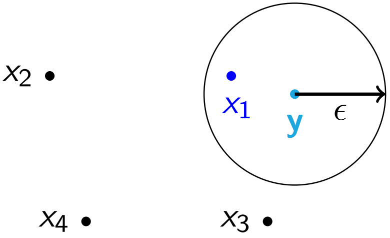
If there is only one codeword in the ball of radius ϵ centred in the received vector y, receiver can decode the message.



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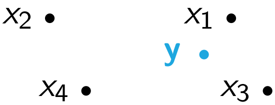

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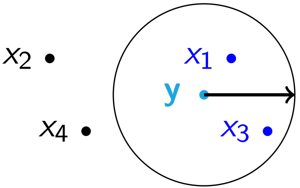
But if there is more than one word in this ball, receiver is confused and can not decode !



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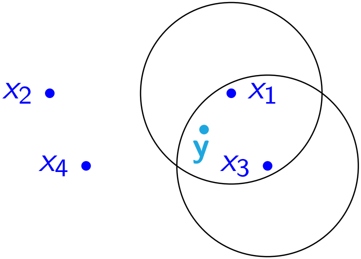
This is equivalent to the fact that the balls of radius ϵ centred in the codewords do not intersect.



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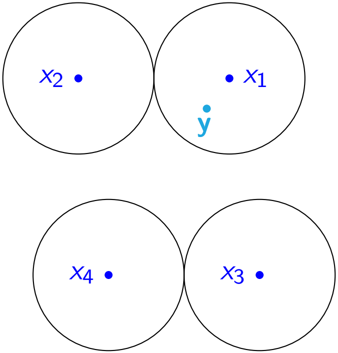
So we would like these balls to be disjoint...



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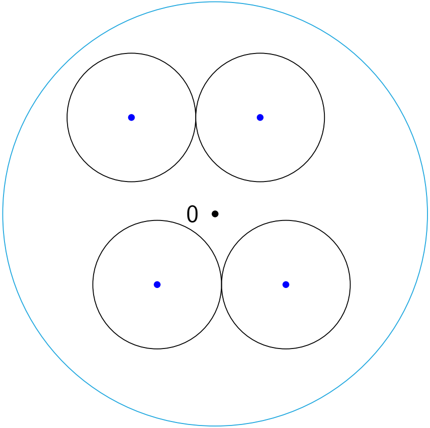
...Keeping as many as possible codewords close to 0.



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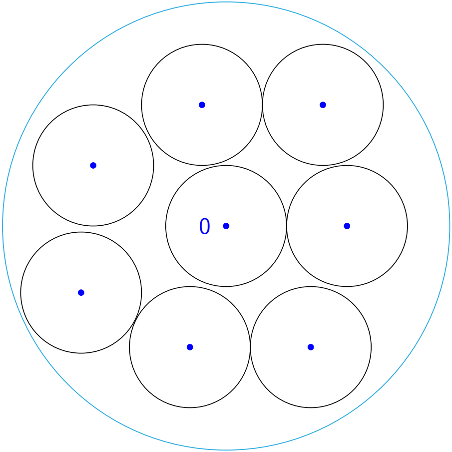

The Sphere Packing Problem

- Finding a good code with respect to this property boils down to finding an arrangement of disjoint spheres having the same radius for which the proportion of space filled is the highest possible.

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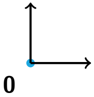
- Sphere packing problem is an old and hard problem of geometry of numbers.
- Euclidean lattices provide a way to approach this problem.

Reminder on Euclidean lattices

A Euclidean lattice Λ in $\mathbb{R}^n$ is the set of all linear combinations with integer coefficients of the elements of a basis B of $\mathbb{R}^n$.



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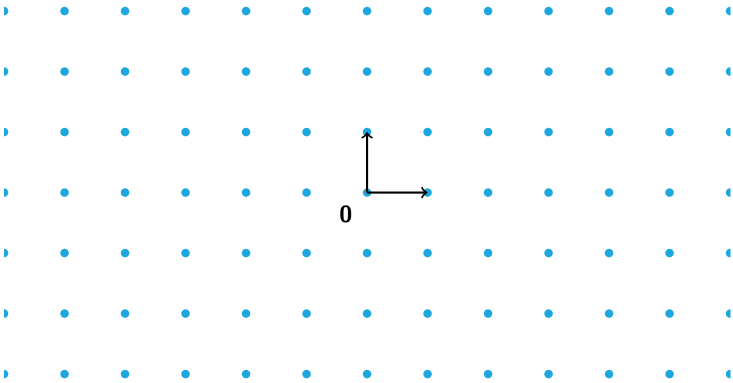

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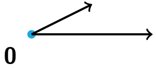

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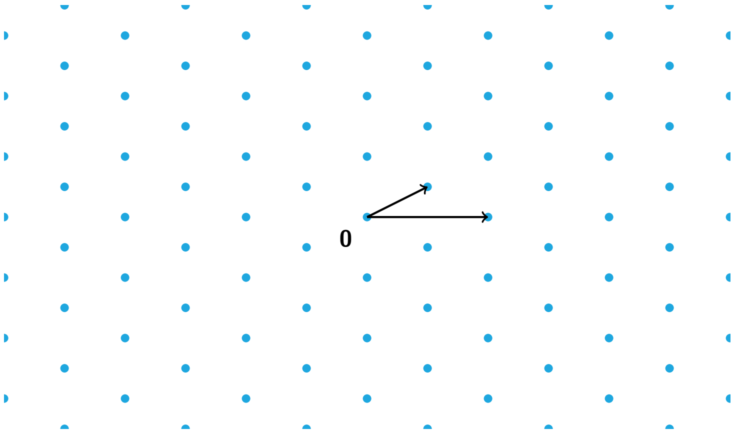
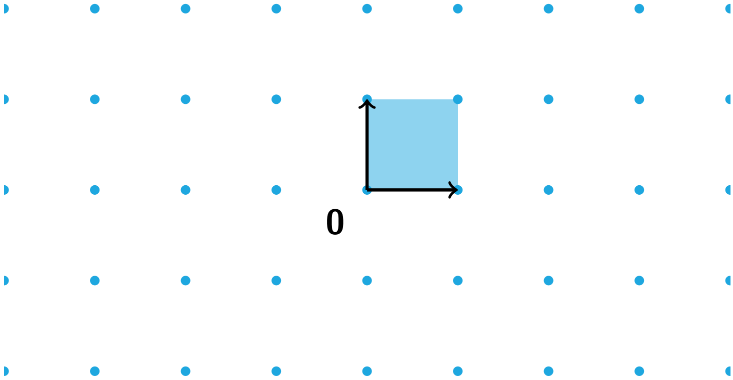


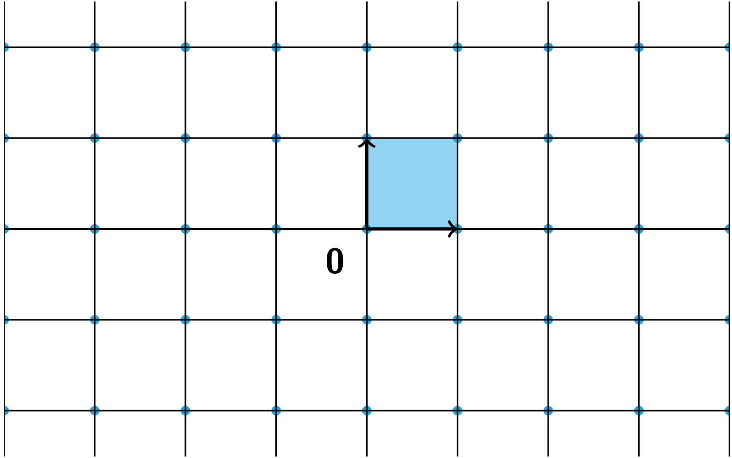
Reminder on Euclidean lattices

Let $\mathcal{P}_B$ the parallelepiped generated by B.



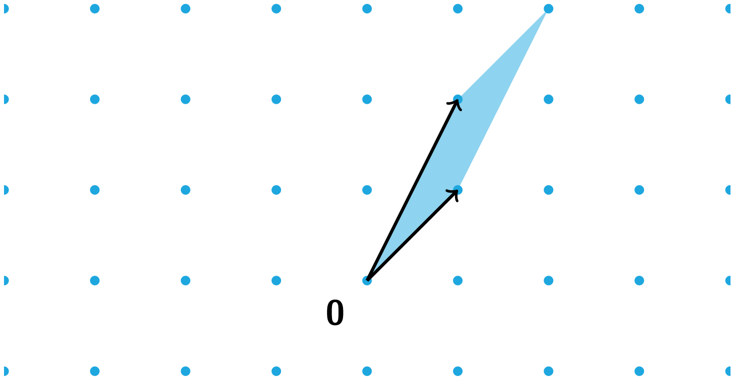
Reminder on Euclidean lattices

Translating $\mathcal{P}_B$ by the points of the lattices, we get a partition of $\mathbb{R}^n$ into equivalent cells . $\mathcal{P}_B$ is called a fundamental region of Λ.



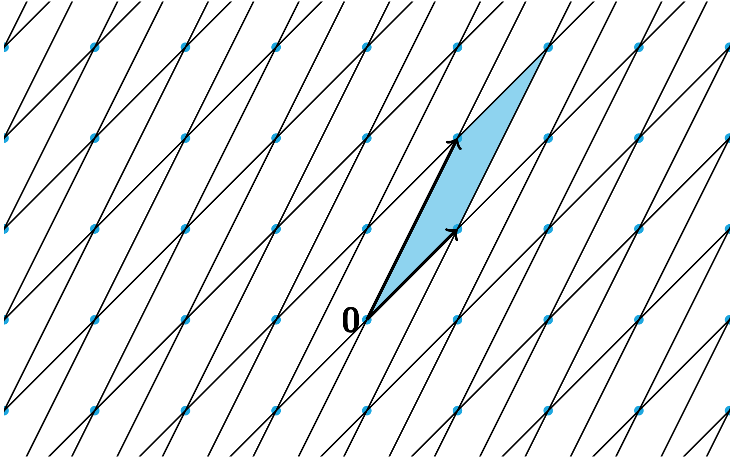
Reminder on Euclidean lattices

This is true for every basis of Λ.



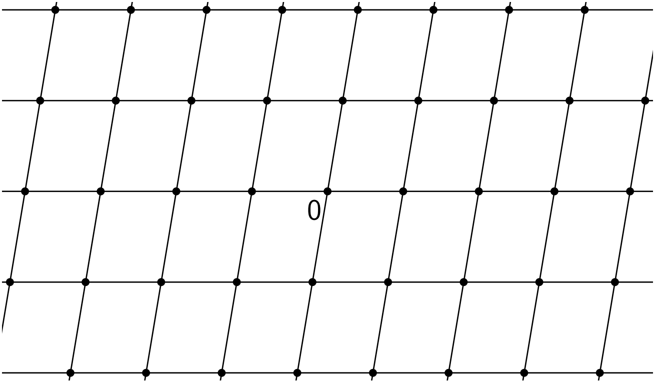
Reminder on Euclidean lattices

Every fundamental region has the same volume. This is the volume of the lattice Λ.



The lattice sphere packing problem

The lattice sphere packing problem consists in finding the biggest proportion of space that can be filled by a collection of disjoint spheres having the same radius, with centers at the points of a lattice Λ.



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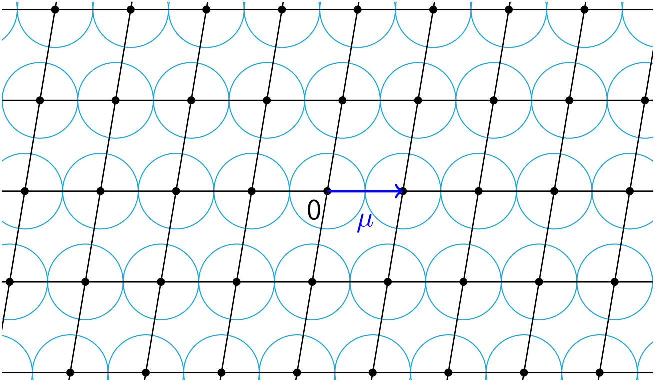
For a given lattice Λ, the best sphere packing associated is given by balls of radius $\mu/2$, where $\mu = \min\{||\lambda||, \lambda \in \Lambda \setminus \{0\}\}$.



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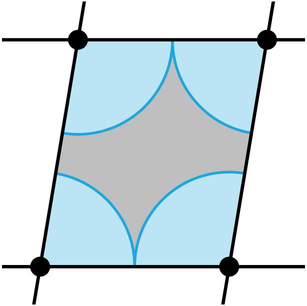
The density of this packing is

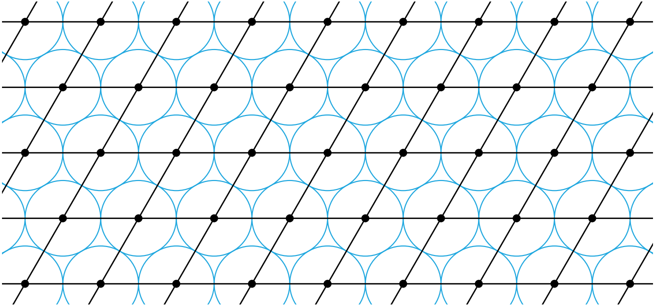
$$\Delta(\Lambda) = \frac{Vol(B(\mu))}{2^n Vol(\Lambda)}$$



Solutions in low dimensions

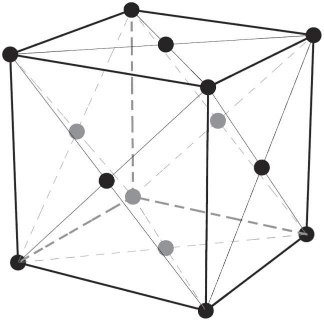
For $n = 1$, the problem is trivial: the best density is 1 !

For $n = 2$, the best packing density is $\frac{\pi\sqrt{3}}{6} \approx 0.9069$, and is given by the hexagonal lattice (Lagrange, 1773, best lattice, Thue, 1892 and Fejes Tóth, 1940, best packing).



Solutions in low dimensions

For $n = 3$, it is the faced-centered cubic lattice which provides the best density $\frac{\pi\sqrt{2}}{6} \approx 0.74048$ (Kepler conjecture, 1611, Gauss, 1832, best lattice, and Hales, 1998, 2014, best packing).



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- For other dimensions, the problem is open.
- Here we are interested in lower bounds for the best packing density Δ_n in dimension n when n goes to infinity.

Summary of results

- Minkowski-Hlawka theorem (stated by Minkowski in 1911, proved by Hlawka in 1943),

$$\Delta_n \geq \frac{2}{2^n}.$$

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- Venkatesh (2013): for all n big enough $\Delta_n \geq \frac{65963n}{2^n}$, and for infinitely many dimensions, $\Delta_n \geq \frac{0.89n \log\log n}{2^n}$.

Some effective results?

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- The best one can do is to find exponential-sized families:
- Rush (1989) gave an "effective" proof of Minkowski-Hlawka theorem, with a family having a size of order $\exp(kn \log n)$.
- Gaborit and Zémor (2006) gave a construction that provides lattices with density higher than $\frac{0.06n}{2^n}$, with a complexity of enumeration of order $\exp(11n \log n)$.

Our result

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We prove an effective version of Venkatesh's theorem:

Theorem

For infinitely many dimension n, one can find a lattice $\Lambda \subset \mathbb{R}^n$ satisfying

$$\Delta(\Lambda) > \frac{0.89 n \log\log n}{2^n}$$

with $\mathcal{O}(\exp(7.8 n \log n))$ binary operations.

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- Basic idea: Let Λ be a lattice in $\mathbb{R}^n$ and $r > 0$.

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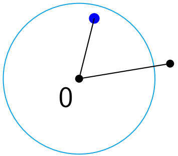

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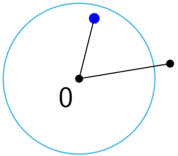

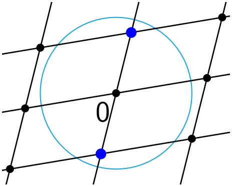

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- So the condition $|B(r) \cap \Lambda \setminus \{0\}| < 2$ is sufficient to conclude $\Delta(\Lambda) \geq \frac{Vol(B(r))}{2^n Vol(\Lambda)}$.

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- For $n = 2\phi(m)$, Venkatesh constructed infinite families of lattices invariant under the action of mth-roots of unity. Taking $m = \prod_{\substack{q \in \mathbb{P} \\ q \leq X}} q$, he optimized the ratio between m and $2\phi(m)$.

Our construction

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- $\mathfrak{P}$ prime ideal of $\mathcal{O}_K = \mathbb{Z}[\zeta_m]$, $F = \mathcal{O}_K / \mathfrak{P} \simeq \mathbb{F}_q$.
- $V = K_{\mathbb{R}}^2$, $\Lambda_0 = \mathcal{O}_K^2$ and

$$\pi : \Lambda_0 \to \Lambda_0 / \mathfrak{P}\Lambda_0 \simeq F^2$$

Definition

Let $\mathcal{C}$ be the set of the $q+1$ F-lines of $\Lambda_0 / \mathfrak{P}\Lambda_0 = F^2$, and $\mathcal{L}_{\mathcal{C}}$ the associated set of lattices of V: $\mathcal{L}_{\mathcal{C}} = \{\pi^{-1}(C), C \in \mathcal{C}\}$.

- Every lattice in $\mathcal{L}_{\mathcal{C}}$ has volume $qVol(\Lambda_0)$ and is invariant under the action of mth-roots of unity.
- The family satisfies, for r and q chosen in a suitable way:

Our construction

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$$\mathbb{E}_{\mathcal{L}_{\mathcal{C}}}[|B(r) \cap \Lambda \setminus \{0\}|] \simeq \frac{Vol(B(r))}{qVol(\Lambda_0)}$$

Results

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Theorem

For every $1 > \varepsilon > 0$, if $\phi(m)^2 m = o(q_m^{\frac{1}{\phi(m)}})$, then for m big enough, the family of lattices $\mathcal{L}_{\mathcal{C}}$ contains a lattice $\Lambda \subset \mathbb{R}^{2\phi(m)}$ satisfying

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- This result is a generalization of Gaborit-Zémor's result: it is valid for a "larger" set of dimensions.
- The action we consider is free: so we have no loss in the constant (1/2 instead of 0.06).

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- For the particular set of dimensions considered by Venkatesh, we obtain the same density...

Corollary

For infinitely many dimensions, $\mathcal{L}_{\mathcal{C}}$ contains a lattice $\Lambda \subset \mathbb{R}^n$ satisfying $\Delta(\Lambda) \geq \frac{0.89 n \log \log n}{2^n}$.

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- ... with finite families of lattices !

Complexity of construction

Let $n = 2\phi(m)$. For every $1 > \varepsilon > 0$, the construction of a lattice $\Lambda \subset \mathbb{R}^n$ satisfying $\Delta(\Lambda) > \frac{(1-\varepsilon)m}{2^{2\phi(m)}}$ requires $\mathcal{O}(\exp(7.8 n \log n))$ binary operations.

Concluding comments and perspectives

- The construction can be adapted in such a way that the lattices of $\mathcal{L}_{\mathcal{C}}$ are symplectic.

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Thank you for your attention