# On the Structure of Large Equidistant Grassmannian Codes

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# Constant intersection Grassmannian Codes

- Denote the $q$-element finite field by $\mathbb{F}_q$. The Grassmannian $\mathcal{G}_q(m, k)$ is the set of all $k$-dimensional vector subspaces of the $m$-dimensional vector space $\mathbb{F}_q^m$.
- A constant dimension subspace code or a Grassmannian code is a subset of $\mathcal{G}_q(m, k)$. Its elements are codewords.
- Projectively, a code $\subseteq \mathcal{G}_q(m, k)$ is a collection of (projective) $(k-1)$-spaces contained in a (projective) $(m-1)$-space $\mathbf{PG(m-1, q)}$.

# Constant intersection Grassmannian Codes

- A Grassmannian code is equidistant or constant distance or constant intersection if every pair of codewords intersect in a subspace of some fixed dimension $t$. It is also called a $t$-intersecting constant dimension code.
- Then say $C \subseteq \mathcal{G}_q(m, k)$ is a **$(k-1, t-1)$-code**. Here we have projective dimension, which equals vector dimension minus 1.
- Assume dimension $m-1$ of ambient projective space $\mathrm{PG}(m-1, q)$, equivalently of $\mathbb{F}_q^m$, is sufficiently large.

# $(k-1, t-1)$-codes

- A *sunflower* is a $(k, t)$-code such that all codewords share a common $t$-space. Thus they are pairwise disjoint outside this $t$-space. On quotienting, equivalent to a partial $(k-t-1)$-spread.
- Let $C \subseteq \mathcal{G}_q(*, k)$ be a $(k-1, t-1)$-code. Etzion and Raviv [*Equidistant codes in the Grassmannian, 2013*] notice that, via a reduction to classical binary equidistant constant weight codes and results of Deza, and, Deza and Frankl:
  If $C$ is not a sunflower then

$$|C| \leq \left(\frac{q^k - q^t}{q-1}\right)^2 + \frac{q^k - q^t}{q-1} + 1.$$

## $(k-1, t-1)$-codes

- If $C$ is not a sunflower then

$$|C| \leq \left(\frac{q^k - q^t}{q-1}\right)^2 + \frac{q^k - q^t}{q-1} + 1.$$

- Conjecture (Deza): If $C$ is not a sunflower then

$$|C| \leq \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q = \frac{q^{k+1}-1}{q-1}.$$

- **Theorem** [*Bartoli, R., Storme, Vandendriessche*]. If $C$ is not a sunflower and $t = 1$ then

$$|C| \leq \left(\frac{q^k - q}{q-1}\right)^2 + \frac{q^k - q}{q-1} + 1 - q^{k-2}.$$

# $(2,0)$-codes

Beutelspacher, Eisfeld, Müller [*On Sets of Planes in Projective Spaces Intersecting Mutually in One Point, 1999*]:

- For projective planes pairwise intersecting in a projective point:
  - the set of points in $\geq 2$ codewords spans a subspace of projective dimension $\leq 6$;
  - there are up to isomorphism only 3 codes $C$ where this projective dimension is 6, all related to the Fano plane.
- For $q \neq 2$ and $|C| \geq 3(q^2 + q + 1)$:
  - $C$ is contained in a Klein quadric in $PG(5, q)$, or
  - is a dual partial spread in $PG(4, q)$, or
  - all codewords have a point in common.

# $(2, 0)$-codes, $q = 2$

- For projective planes pairwise intersecting in a projective point, for $q = 2$:
  Deza's Conjecture: If $C$ is not a sunflower then

$$|C| \leq 15.$$

- Bartoli and Pavese [*A note on equidistant subspace codes, 2015*] disproved it and found a code with

$$|C| = 21,$$

with a unique such example.

## $(n, n-t)$-codes

- A code of projective $n$-spaces pairwise intersecting exactly in an $(n-t)$-space.
- An *intersection point* is a point contained in $\geq 2$ codewords.
- The *base* $\mathcal{B}(S)$ of a codeword $S$ is the span of intersection points contained in it.
- Extending the definition of a code

$$C \subseteq \mathcal{G}_q(*, n)$$

to a code

$$C \subseteq \mathcal{G}_q(*, n) \cup \mathcal{G}_q(*, n-1) \cup \dots,$$

we may replace each codeword by its base.

# Primitive $(n, n-t)$-codes

- If the ambient projective space is $(2n+1-\delta)$-dimensional, the dual of an $(n, n-t)$-code is an $(n-\delta, n-\delta-t)$-code.
- If $\exists$ a point contained in all codewords then, upon quotienting by it, we have an $(n-1, n-1-t)$-code.
- An $(\leq n, n-t)$-code is a collection of **at-most-$n$-spaces** pairwise intersecting exactly in an $(n-t)$-space.
- An $(n, n-t)$-code $C$ is *primitive* (old definition by Eisfeld) if
  1. all $\mathcal{B}(S) := \langle S \cap T : T \in C \backslash \{S\} \rangle$, where $S \in C$, are $n$-dimensional;
  2. ambient space has dimension at least $2n+1$.
  3. there is no point contained in all codewords;
  4. ambient space is the span of all codewords;
  5. $S = \mathcal{B}(S)$ for all $S \in C$.

# New primitivity

- To make primitivity definition self-dual, should add:

$$6. \text{ For all codewords } S \in C: \quad S = \bigcap_{T \in C \setminus \{S\}} \langle S, T \rangle.$$

- So, say an $(n, n-t)$-code $C$ is *'new' primitive* (new definition by us) if 1. - 6. hold.
- Conditions 3. and 4. are dual.
  Conditions 5. and 6. are dual.
- Condition 2. allows induction on $n$ by dualisation.
- Conditions 3. and 4. allow induction on $n$ by quotienting.
- Definition remains self-dual if generalised to codewords of several dimensions and several intersection dimensions, i.e. if we keep 3. - 6.

## $(n, n-t)$-codes with small $t$

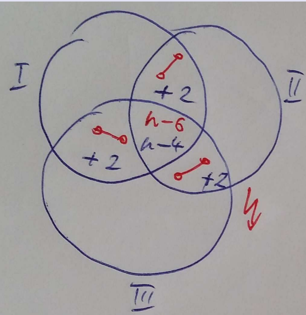
- For $t = 0$ we have $|C| = 1$.
- For $t = 1$: for an $(n, n-1)$-code, equivalently, intersections are **at least** dimension $n-1$.
- By geometric Erdős-Ko-Rado: then all codewords
  1) share a common $(n-1)$-space, i.e. they form a *sunflower*, or,
  2) are contained in a common $(n+1)$-space (since any codeword $S$ is contained in $\langle S_1, S_2 \rangle$ for some codewords $S_1, S_2$ such that $S_1 \cap S_2 \not\subseteq S$), i.e. they form a *ball*.
- Thus $(n, n-1)$-codes are classified.

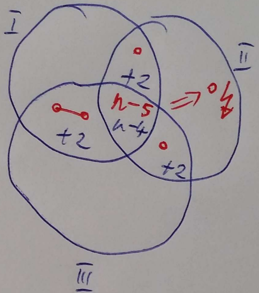
# Classifying $(n, n-2)$-codes

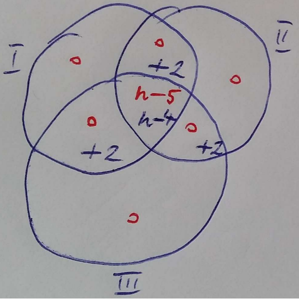
- If $\exists$ a point in common in all codewords of an $(n, n-2)$-code, quotient by it to get an $(n-1, n-3)$-code. Such codes are thus classified by induction on $n$.
- We may assume $\langle S : S \in C \rangle$ is the ambient space. (Intersection properties do not change; otherwise, in the dual code there is a point in common in all codewords.)
- If ambient space dimension is $2n+1-\delta$ then the **dual** of
  - an $(n, n-t)$-code is an $(n-\delta, n-t-\delta)$-code;
  - an $(\leq n, n-t)$-code is an $(\geq n-\delta, n-t-\delta)$-code.

# Classifying $(n, n-2)$-codes

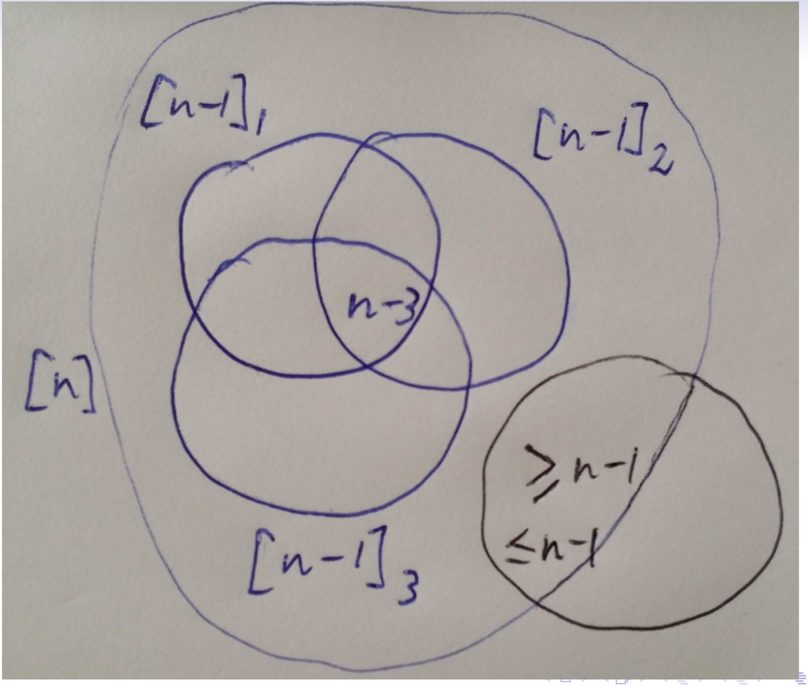
- Remember: An $(n, n-t)$-code $C$ is equivalent to an $(\leq n, n-t)$-code $C' = \{\mathcal{B}(S) \mid S \in C\}$.
- Say *dimension* of $S \in C$ is $\dim(\mathcal{B}(S))$.
- For $\geq 2$ codewords, the dimension of each codeword is $n-2$, $n-1$ or $n$. If a dimension is $n-2$, the code $C$ is a sunflower; so let codeword dimensions be $n-1$ or $n$.

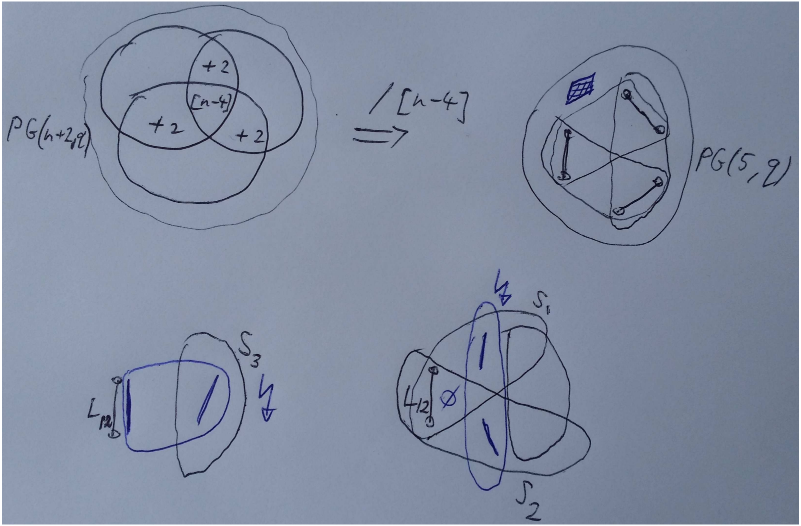


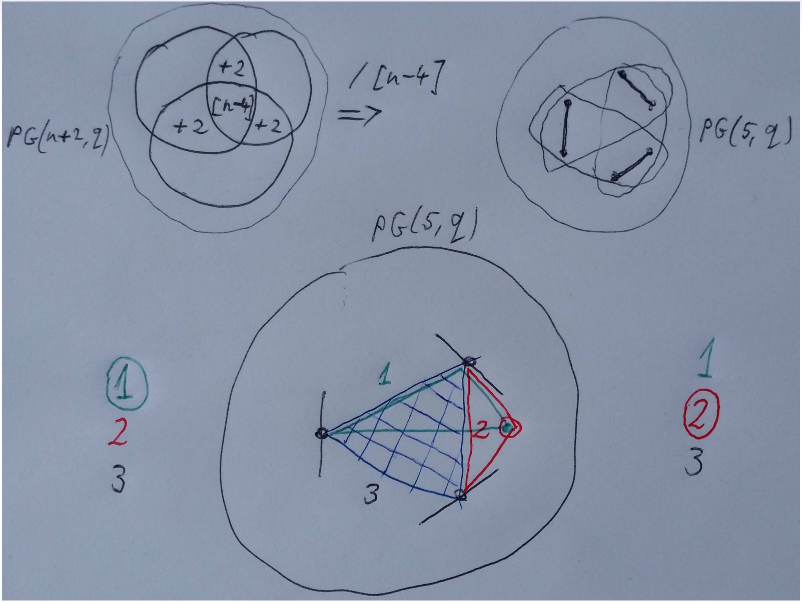



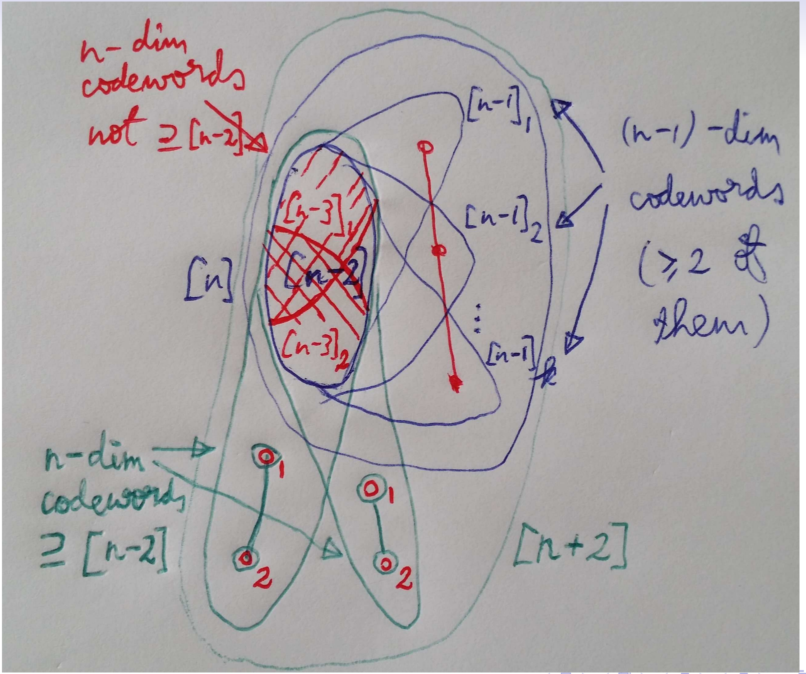




A red codeword intersecting $\langle \overset{I}{O}, \overset{II}{O}, \overset{III}{O} \rangle$ in dimension $n-1$.

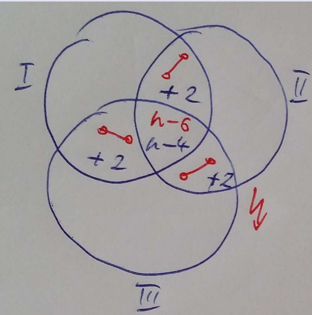

[n-1]₁
[n-1]₂
n-3
[n]
≥n-1
≤n-1
[n-1]₃

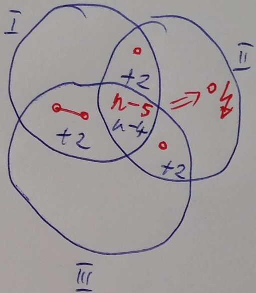

+2
[n-4]
+2
+2
PG(n+2,q)
/[n-4]
PG(5,q)
S3
L12
S1
L12
∅
S2

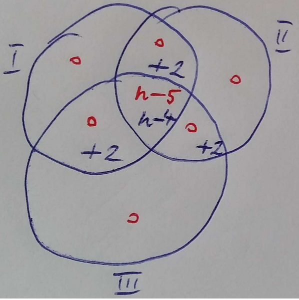

PG(n+2,q)
+2
[n-4]
+2
+2
/ [n-4]
=>
PG(5,q)
PG(5,q)
1
2
3
1
2
3
1
2
3



n-dim codewords not ⊇ [n-2]
[n-1]_1
[n-1]_2
[n-1]_k
(n-1)-dim codewords (≥ 2 of them)
[n-3]_1
[n-2]
[n-3]_2
[n]
n-dim codewords ⊇ [n-2]
1
2
1
2
[n+2]




A red codeword intersecting $\langle \overset{I}{0}, \overset{II}{0}, \overset{III}{0} \rangle$ in dimension $n-1$.

Thank you!