Binary Locally Repairable Codes with High Availability via Anticodes

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Joint work with Alexander Zeh

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Outline

- Codes with Locality and Availability
  - Motivation: Distributed Storage
  - Known Bounds
- AntiCodes and AntiCode-Based Construction
- Our Results:
  - Optimal Codes with Locality and Availability
- Summary and Outlook

Locality

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  - Erasure codes which allow **local** correction of an erasure (using a **small** number of code symbols)

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  - Erasure codes which allow **local** correction of an erasure (using a **small** number of code symbols)
- The ith code symbol $c_i, 1 \leq i \leq n$ of an $[n, k, d]$ code C is said to have **locality $\boldsymbol{r}$** if c_i can be recovered by accessing at most r other code symbols.
- An $[n, k, d]$ code C is called **$\boldsymbol{r}$-LRC** if all its symbols have locality r.

Availability

- **Codes with availability**
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- The ith code symbol $c_i, 1 \leq i \leq n$ of an $[n, k, d]$ code C is said to have **locality $\boldsymbol{r}$** and **availability $\boldsymbol{t}$** if c_i can be recovered from t disjoint sets of other code symbols, (called repair sets), where ∀|repair set|$\leq r$.
- An $[n, k, d]$ code C is called **$(\boldsymbol{r}, \boldsymbol{t})$-LRC** if all its symbols have locality r and availability t.
- If $t = 1$ then $(r, 1)$-LRC is an r-LRC .

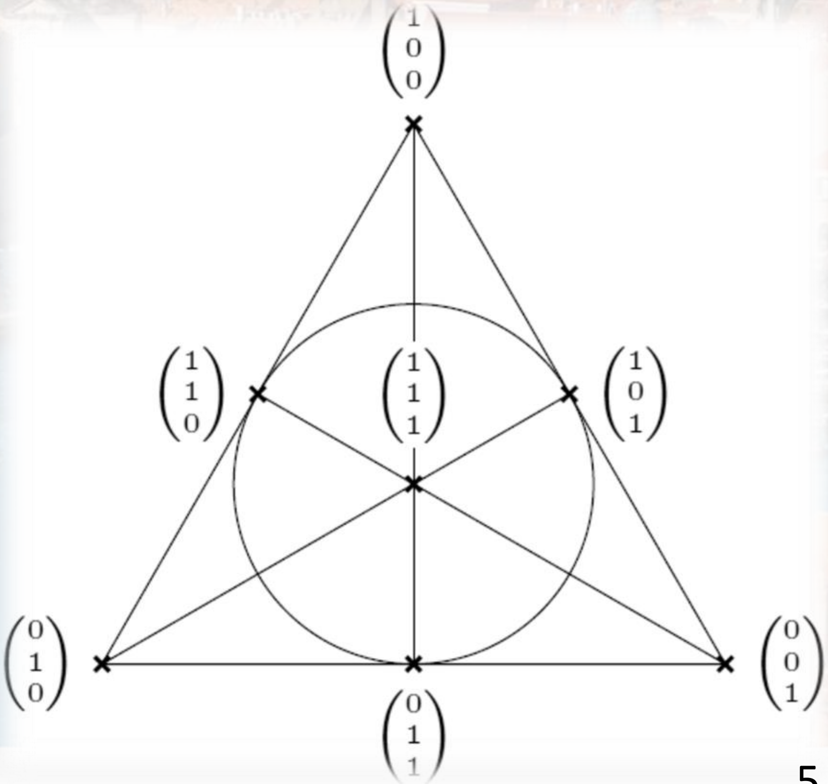
(r,t)-LRC: generator matrix

- Let $G = (g_1|g_2|\ldots|g_n)$ be a generator matrix of an $[n,k,d]$ code C. The ith symbol of C has locality r and availability t if there exist t sets $R_1^i, R_2^i, \ldots, R_t^i \subseteq [n]\backslash\{i\}$ s.t.
- $R_j^i \cap R_s^i = \emptyset$, j $\neq s \in [t]$
- $|R_s^i| \leq r, s \in [t]$
- $g_i \in \text{span}\left\{g_j\right\}_{j\in R_s^i}, s \in [t]$

Example

- Binary [7,3,4] Simplex code S_3

- $G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$



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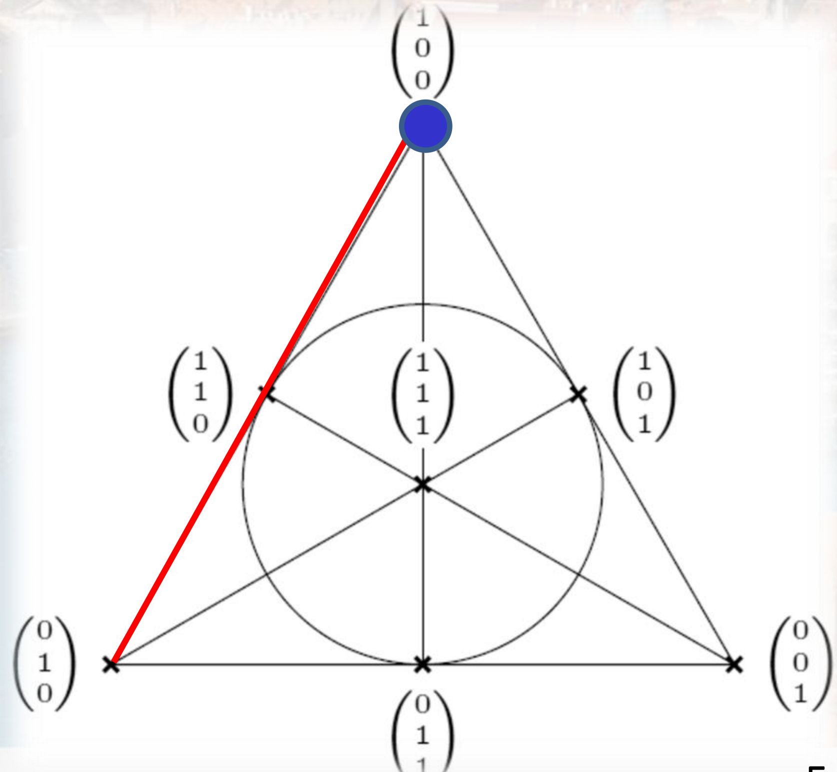

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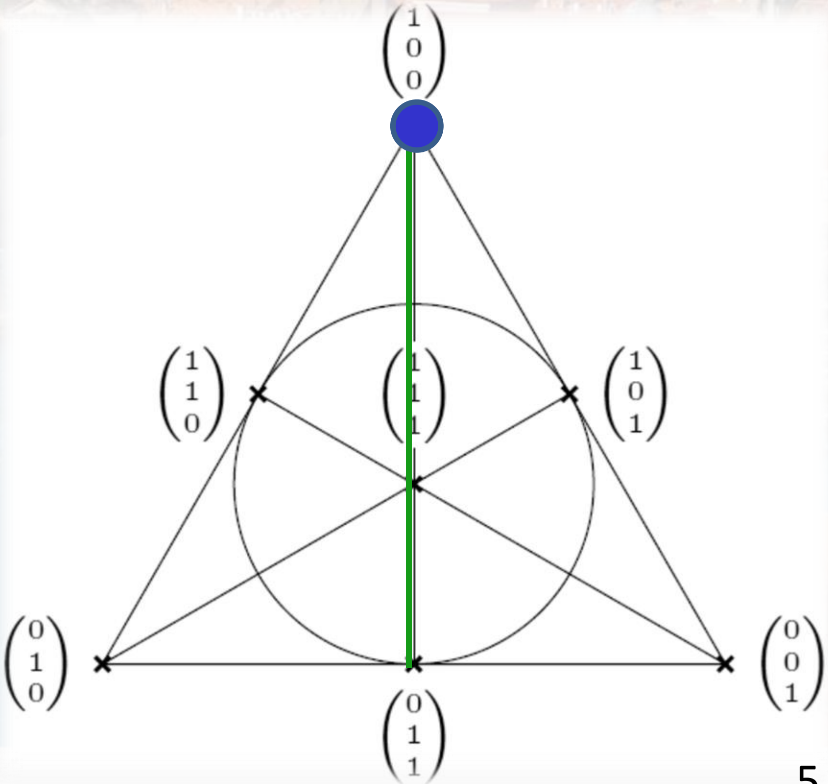

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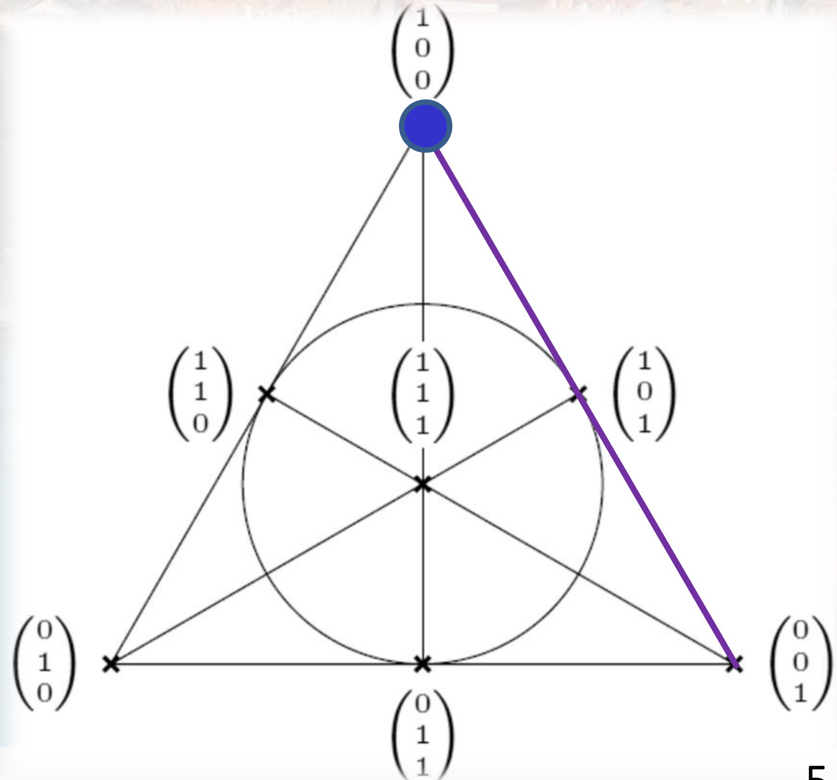

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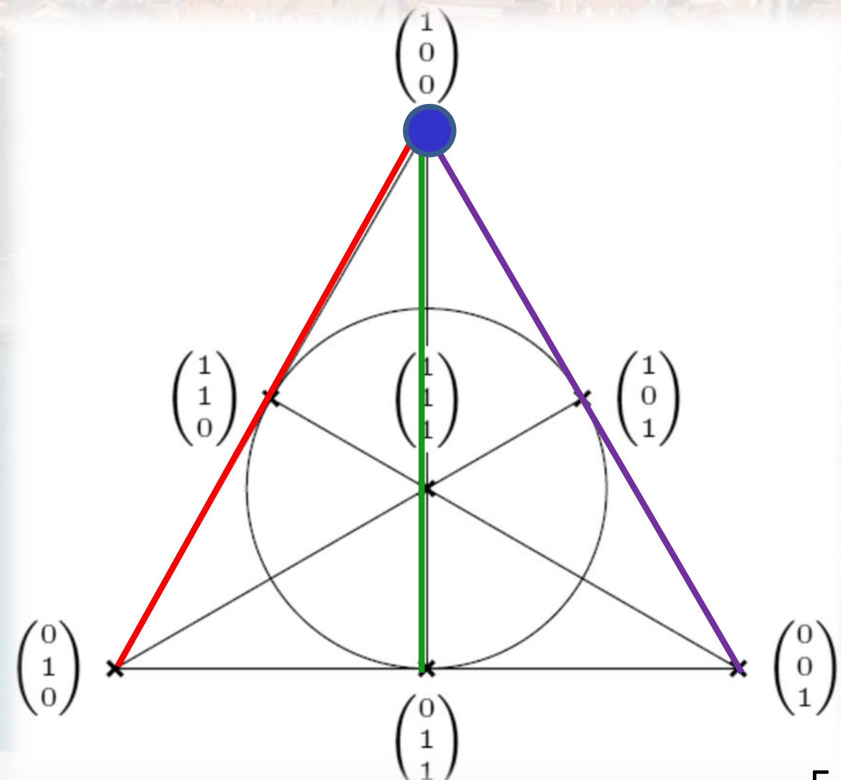

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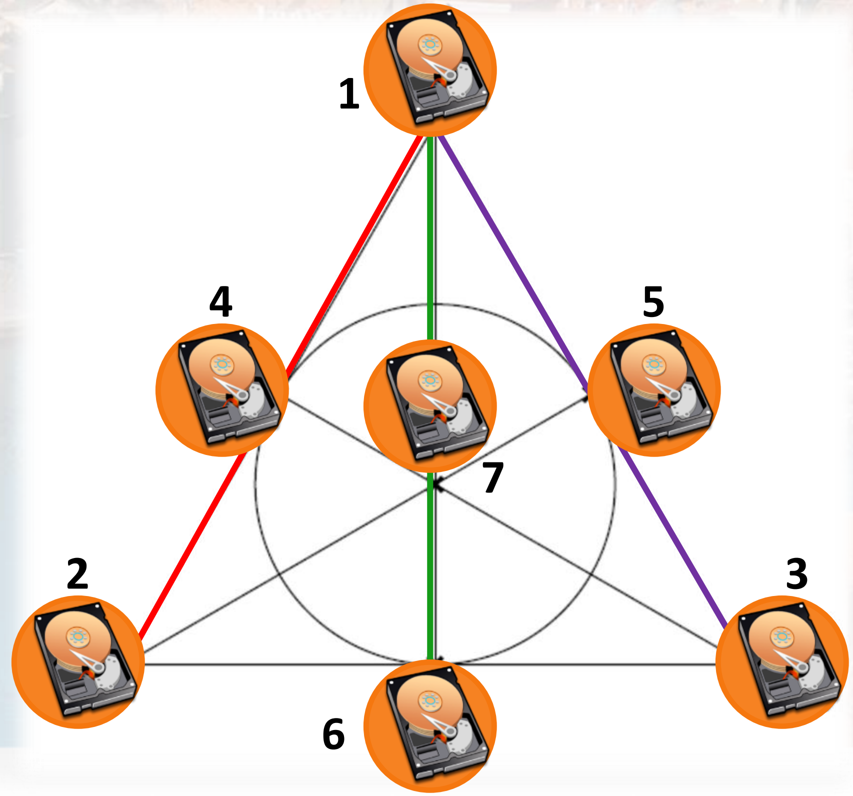
Distributed Storage System



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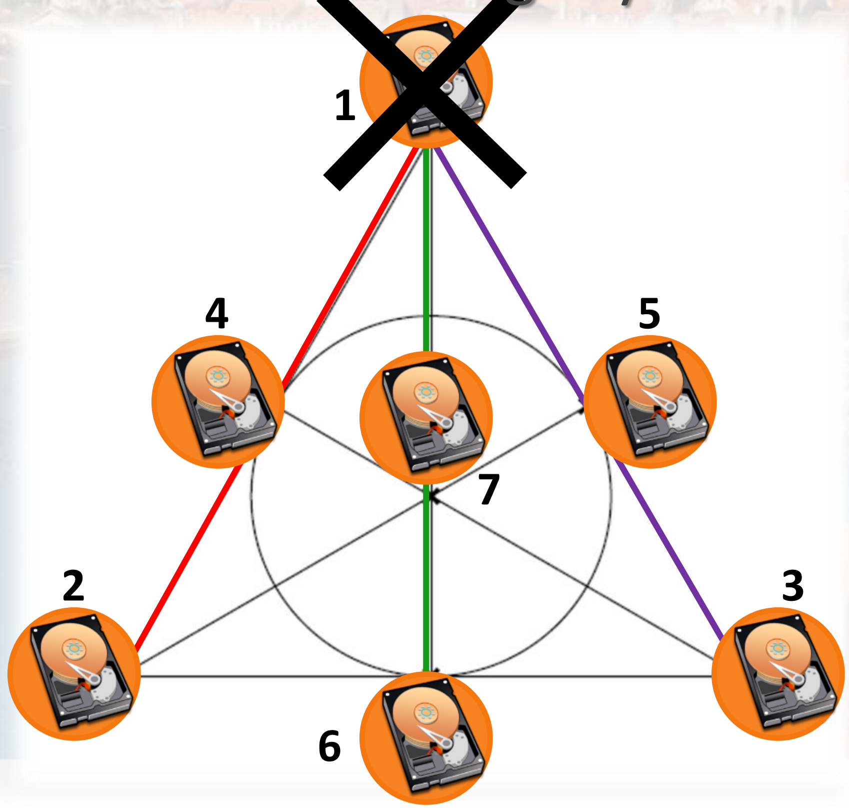

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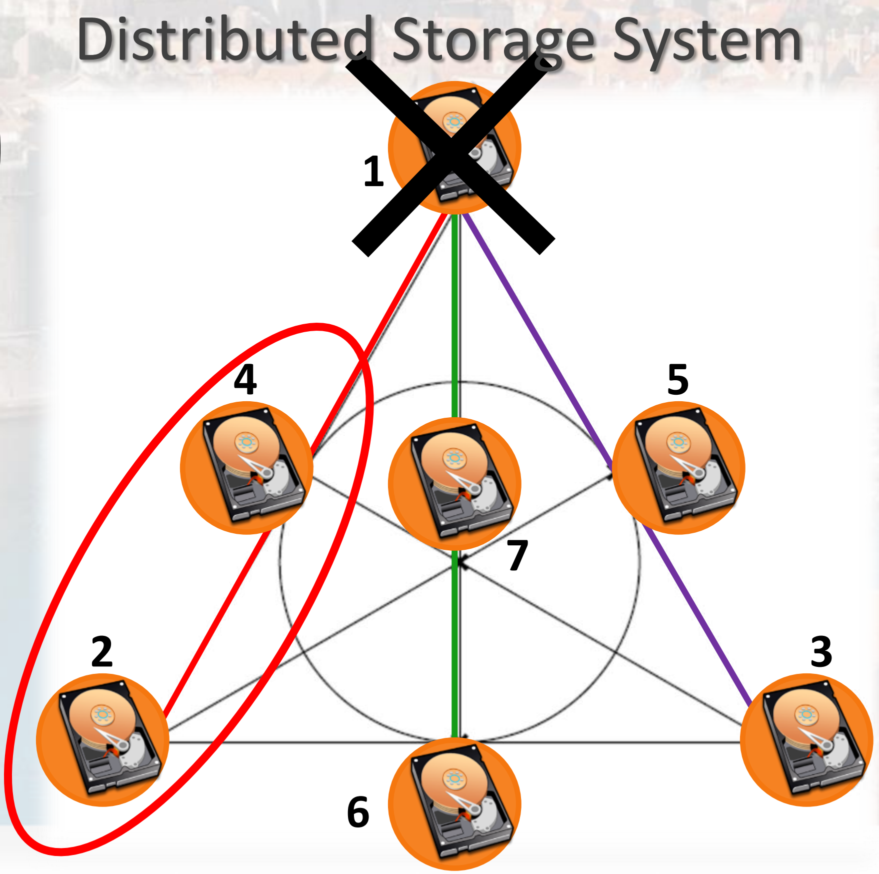

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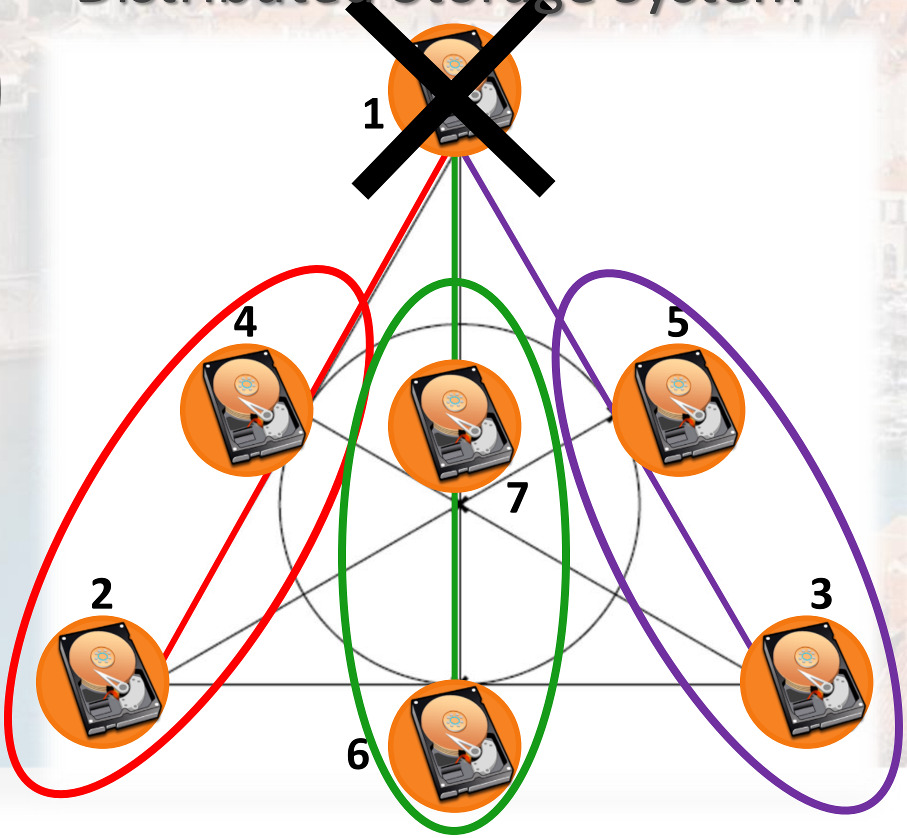

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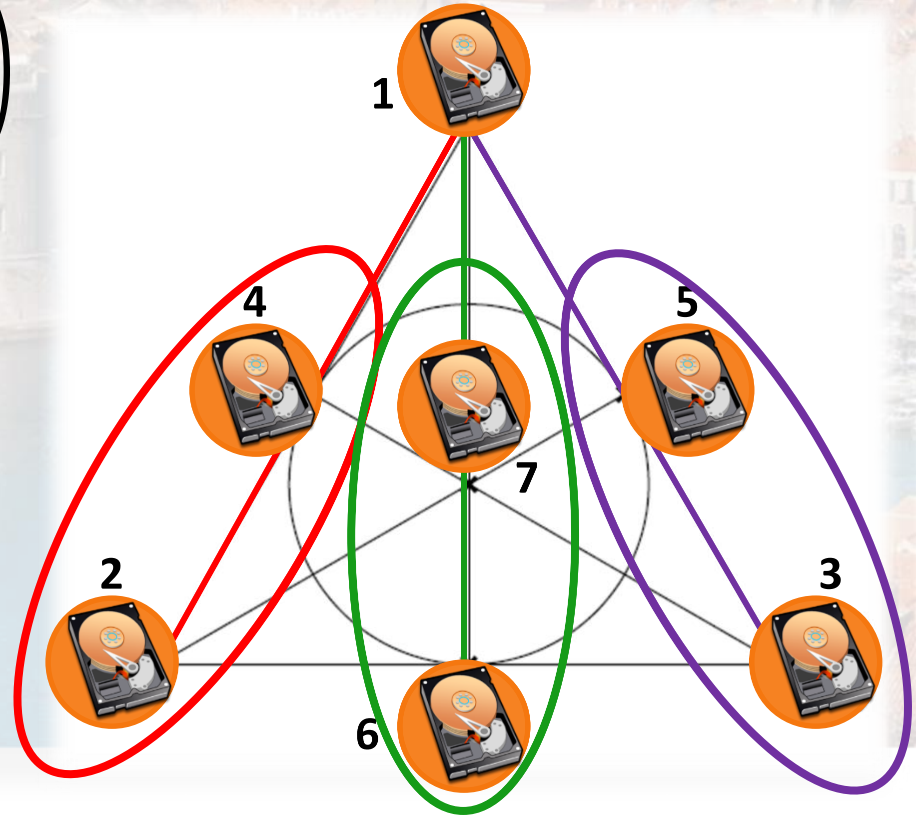

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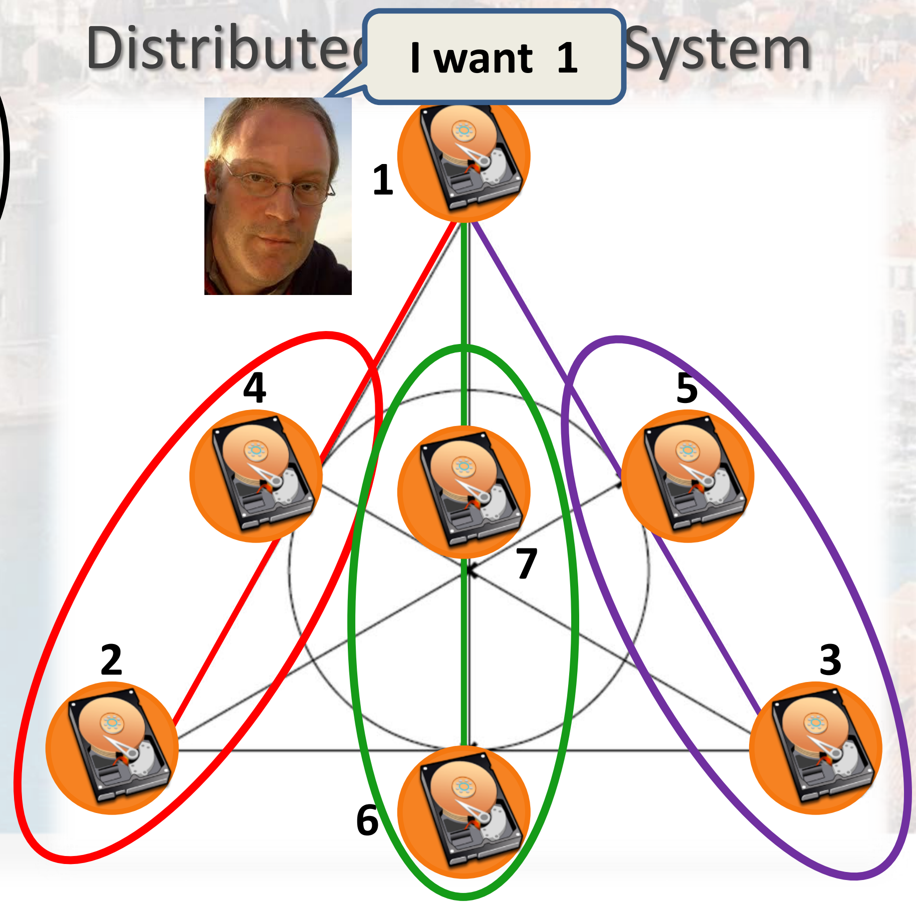

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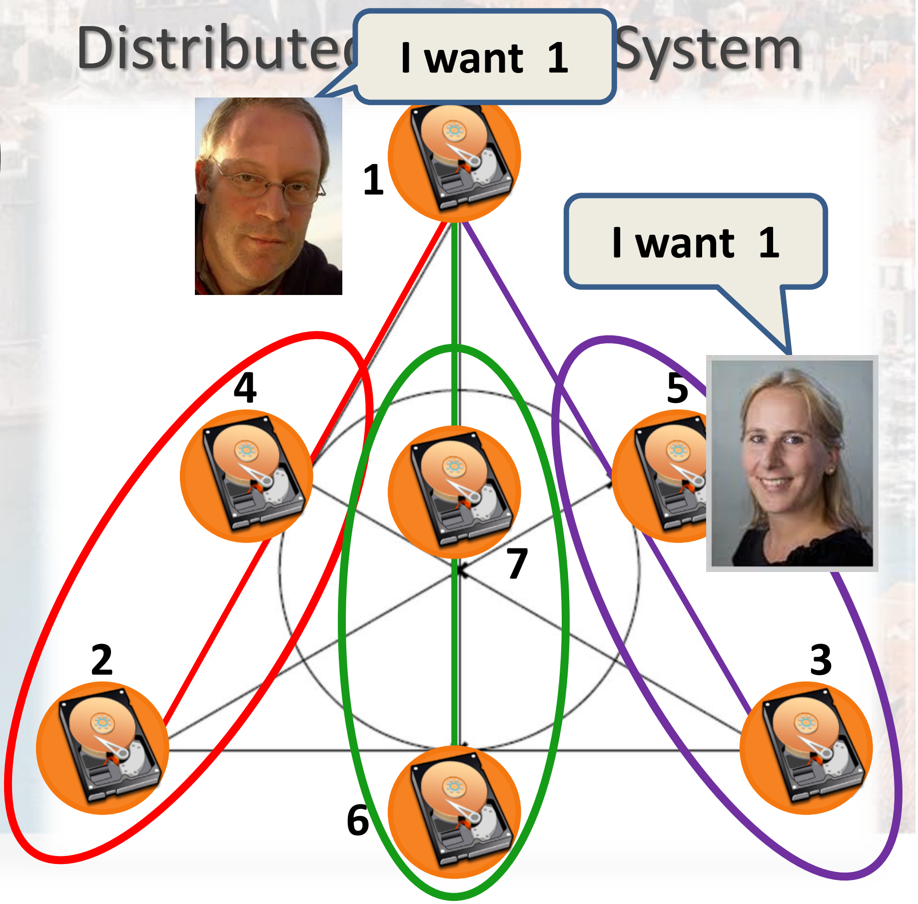

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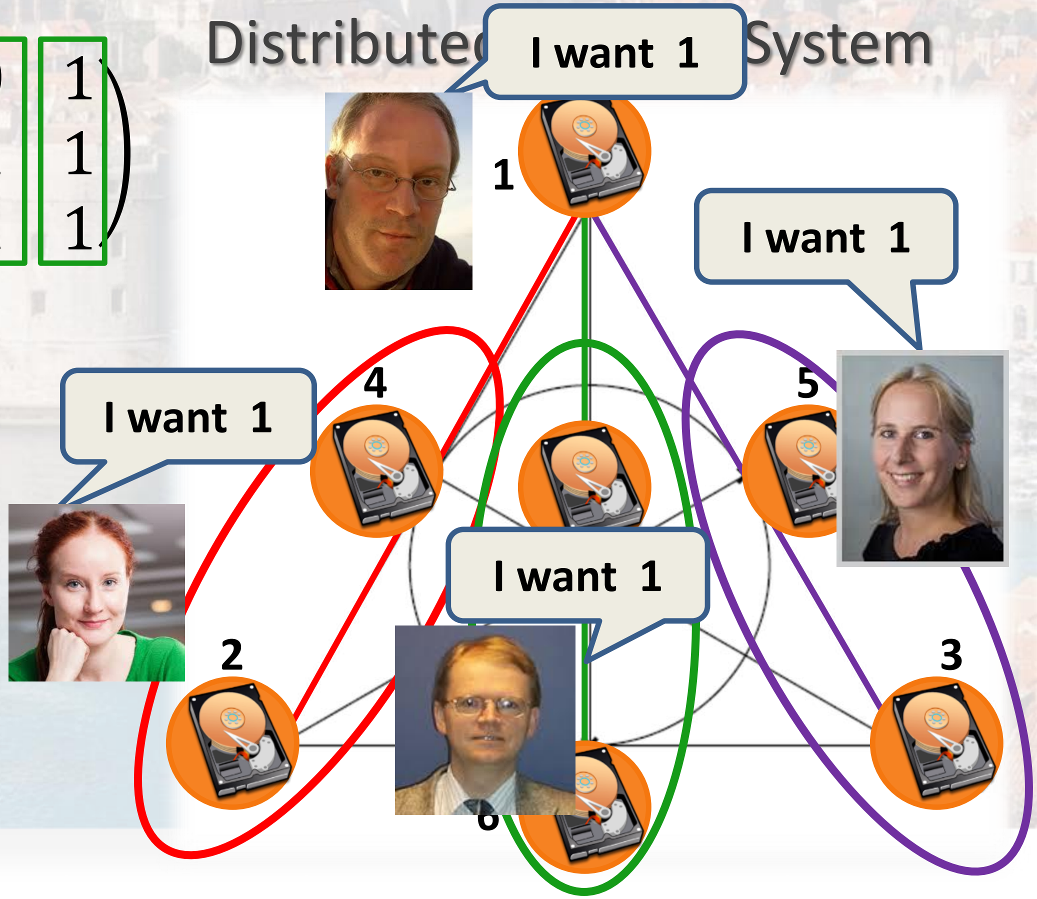

Simplex Codes

- Binary $[2^m - 1, m, 2^{m-1}]$ Simplex code S_m

> Locality $r = 2$
> Availability $t = 2^{m-1} - 1$

- Recall: The columns of the generator matrix G_m of S_m are all distinct nonzero vectors of $\mathbb{F}_2^m$.

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Bounds

<u>Theorem 1</u> [GHSY12]. Let an $[n, k, d]$ code C be an r-LRC. The rate and the minimum distance of C satisfy

$$\frac{k}{n} \leq \frac{r}{r+1}, \qquad d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2$$

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<u>Theorem 2</u> [RPDV14, TB14]. Let an $[n, k, d]$ code C be an (r, t)-LRC. The rate and the minimum distance of C satisfy

$$\frac{k}{n} \leq \frac{1}{\prod_{i=1}^{t}(1 + \frac{1}{jr})}, \qquad d \leq n - k - \left\lceil \frac{t(k-1)+1}{t(r-1)+1} \right\rceil + 2$$

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All the known codes that attain the bounds on the minimum distance are defined over **large** alphabets.

Alphabet-Dependent Bound

Theorem 3 [CM15]. Let an $[n, k, d]$ code C be an r-LRC over $\mathbb{F}_q$. The dimension of C satisfies

$$k \leq \min_{i \in \mathbb{Z}^+} \left\{ ir + k_{opt}^{q}(n - i(r+1), d) \right\},$$

where $k_{opt}^{q}(n, d)$ is the largest possible dimension of a code of length n, for a given alphabet size q and a given minimum distance d.

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- Note that the rate of an (r, t)-LRC is **at most** the rate of an r-LRC with the same parameters r, n, d.

  => The bound of Theorem 3 applies for an (r, t)-LRC.

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- A code which attains this bound will be called ***CM-optimal***.
- Example: binary Simplex code is CM-optimal.

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Our Goal: Construct new codes which are CM-optimal.

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Anticode-Based Construction

- Proposed by P. Farell in 1970s to obtain optimal codes which attain Griesmer bound

- Based on deleting certain columns from the generator matrix of the Simplex code, where the deleted columns form an anticode

Anticodes

- A binary linear $[n, k, \delta]$ **anticode** $\mathcal{A}$ is a set of codewords in $\mathbb{F}_2^n$ with the **maximum** distance δ.
- Distance of **zero** between codewords is allowed.
- Let $G_{\mathcal{A}}$ be a $k \times n$ generator matrix of $\mathcal{A}$. If $\mathrm{rk}(\mathcal{A}) = \gamma$ then each codeword occurs $2^{k-\gamma}$ times in $\mathcal{A}$.
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Example:

A $[3,3,2]$ anticode $\mathcal{A}$ generated by $G_{\mathcal{A}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is given by

$$\mathcal{A} = \{(000), (110), (101), (011), (011), (101), (110), (000)\}$$

Anticode-Based Construction

- Let S_m be a $[2^m - 1, m, 2^{m-1}]$ Simplex code with a generator matrix G_m.
- Let $\mathcal{A}$ be an $[n, k, \delta]$ anticode with a generator matrix $G_{\mathcal{A}}$.
- Then $G = G_m \setminus G_{\mathcal{A}}$, the matrix obtained by deleting n columns of $G_{\mathcal{A}}$ from G_m, is a generator matrix of a $[2^m - 1 - n, \leq m, 2^{m-1} - \delta]$ code.

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Example:

$$G_4 = \begin{pmatrix} 1&0&0&0&1&1&1&0&0&0&1&1&1&0&1 \\ 0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&0&1&0&0&1&0&1&0&1&1&0&1&1&1 \\ 0&0&0&1&0&0&1&0&1&1&0&1&1&1&1 \end{pmatrix}, G_{\mathcal{A}} = \begin{pmatrix} 1&1&0 \\ 1&0&1 \\ 0&1&1 \end{pmatrix}$$

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generates a $[12,4,6]$ code (which attains Griesmer bound)

Our Codes

- **<u>Idea</u>:** To apply anticode-based construction with *good* anticodes which allow to achieve
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Our Codes

- **<u>Idea</u>:** To apply anticode-based construction with *good* anticodes which allow to achieve
  - Small locality
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  - CM-optimality\Griesmer-optimality\both
- We construct 4 families of such anticodes
- => 4 families of optimal codes with small locality and high availability

AntiCode #1

- Let $\mathcal{A}_{s,2}$ be an anticode such that all weight-2 vectors of length s form the columns of $G_{\mathcal{A}_{s,2}}$.

  Then $\mathcal{A}_{s,2}$ is an $[\binom{s}{2}, \mathrm{s}, \delta]$ with $\delta = \lfloor s^2/4 \rfloor$

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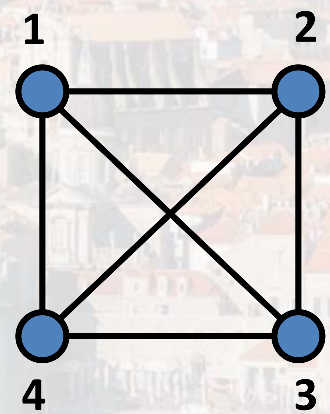
- ***<u>Proof</u>***:
  - length: trivial
  - Maximum distance δ :

  Note that $G_{\mathcal{A}_{s,2}}$= incidence matrix of a complete graph K_s.
  Then δ is equal to the size of the maximum cut between a vertex set of size i and its complement, for $1 \leq i \leq s$.
  Such a cut is of size $\lfloor s^2/4 \rfloor$.

AntiCode #1

- **<u>Example</u>:**

$$G_{\mathcal{A}_{s,2}} = \begin{pmatrix} 1\ 1\ 1\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1\ 1\ 0 \\ 0\ 1\ 0\ 1\ 0\ 1 \\ 0\ 0\ 1\ 0\ 1\ 1 \end{pmatrix} \Leftrightarrow$$



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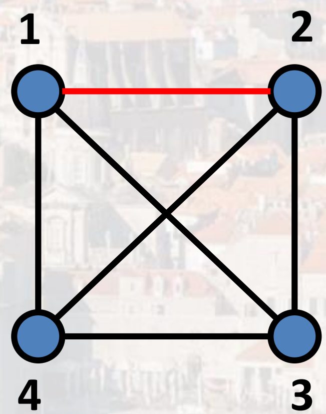

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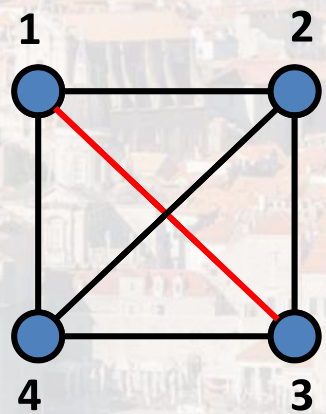

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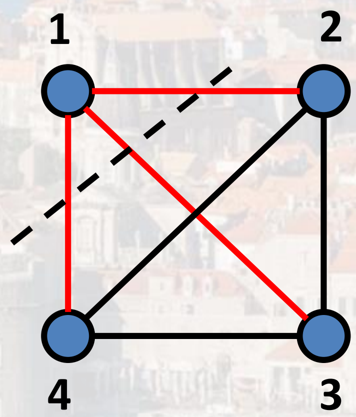

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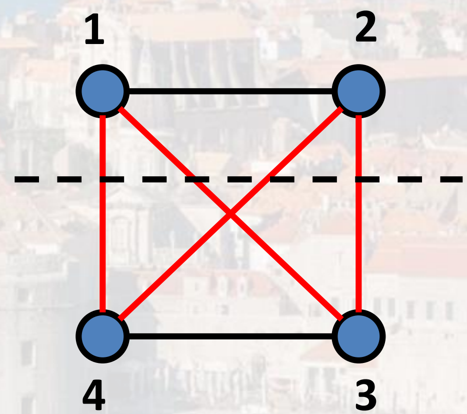

Parameters of Code C_I

- **Theorem 4.** Let
  - G_m: $[2^m - 1, m, 2^{m-1}]$ Simplex code S_m
  - $G_{\mathcal{A}_{s,2}}$: $[\binom{s}{2}, s, \lfloor s^2/4 \rfloor]$ anticode $\mathcal{A}_{s,2}, s \leq m$

  Then $G_I = G_m \setminus G_{\mathcal{A}_{s,2}}$ generates an

  $[2^m - \binom{s}{2} - 1, m, 2^{m-1} - \lfloor s^2/4 \rfloor]$ (r, t)-LRC C_I with

  locality $r = 2$ and availability $t = 2^{m-1} - \binom{s}{2} - 1$.

Parameters of Code C_I

- **Theorem 4.** Let
  - G_m: $[2^m - 1, m, 2^{m-1}]$ Simplex code S_m
  - $G_{\mathcal{A}_{s,2}}$: $[\binom{s}{2}, \mathrm{s}, \lfloor s^2/4 \rfloor]$ anticode $\mathcal{A}_{s,2}, s \leq m$

  Then $G_I = G_m \setminus G_{\mathcal{A}_{s,2}}$ generates an

  $[2^m - \binom{s}{2} - 1, m, 2^{m-1} - \lfloor s^2/4 \rfloor]$ (r, t)-LRC C_I with

  locality $r = 2$ and availability $t = 2^{m-1} - \binom{s}{2} - 1.$

Proof (locality+availability):

Given a column g of G_I, there are $t_m = 2^{m-1} - 1$ two-dimensional subspaces which contain g from which we remove at most $|\mathcal{A}_{s,2}| = \binom{s}{2}$ whose columns belong to $G_{\mathcal{A}_{s,2}}$.

Optimality of C_I

- **Theorem 4.** Let
  - G_m: $[2^m - 1, m, 2^{m-1}]$ Simplex code S_m
  - $G_{\mathcal{A}_{s,2}}$: $[\binom{s}{2}, s, \lfloor s^2/4 \rfloor]$ anticode $\mathcal{A}_{s,2}$, $s \leq m$

  Then $G_I = G_m \setminus G_{\mathcal{A}_{s,2}}$ generates an

  $[2^m - \binom{s}{2} - 1, m, 2^{m-1} - \lfloor s^2/4 \rfloor]$ (r,t)-LRC C_I with

  locality $r = 2$ and availability $t = 2^{m-1} - \binom{s}{2} - 1$.

Optimality of C_I:
- For $s \in \{3,4,5\}$ is CM-optimal
- For $s \in \{3,4\}$ is Griesmer-optimal

Anticode #2

- Let $\mathcal{A}_{s;[2,s-1]}$ be an anticode with the generator matrix $G_{\mathcal{A}}$:
  - The columns of $G_{\mathcal{A}}$ are all vectors in $\mathbb{F}_2^s$ with weights in $\{2,3,\dots,s-1\}$
- $\mathcal{A}_{s;[2,s-1]}$ is a $[2^s - s - 2, s, 2^{s-1} - 2]$ anticode

Parameters of Code C_{II}

- Let $\mathcal{A}_{s;[2,s-1]}$ be an anticode with the generator matrix $G_{\mathcal{A}}$:
  - The columns of $G_{\mathcal{A}}$ are all vectors in $\mathbb{F}_2^s$ with weights in $\{2,3,\ldots,s-1\}$
- $\mathcal{A}_{s;[2,s-1]}$ is a $[2^s - s - 2, s, 2^{s-1} - 2]$ anticode
- **<u>Theorem 5.</u>** Let
  - G_m: $[2^m - 1, m, 2^{m-1}]$ Simplex code S_m
  - $G_{\mathcal{A}}$: $[2^s - s - 2, s, 2^{s-1} - 2]$ anticode $\mathcal{A}_{s;[2,s-1]}$, $s \leq m - 1$

  Then $G_{II} = G_m \setminus G_{\mathcal{A}}$ generates an

$$[2^m - 2^s + s + 1, m, 2^{m-1} - 2^{s-1} + 2]$$

$(2, t)$-LRC C_{II} with locality 2 and availability $t = 2^{m-1} - 2^s + \mathrm{s} + 1$.

Optimality of C_{II}

- Let $\mathcal{A}_{s;[2,s-1]}$ be an anticode with the generator matrix $G_{\mathcal{A}}$:
  – The columns of $G_{\mathcal{A}}$ are all vectors in $\mathbb{F}_2^s$ with weights in $\{2,3,\ldots,s-1\}$
- $\mathcal{A}_{s;[2,s-1]}$ is a $[2^s - s - 2, s, 2^{s-1} - 2]$ anticode
- **Theorem 5.** Let
  – G_m: $[2^m - 1, m, 2^{m-1}]$ Simplex code S_m
  – $G_{\mathcal{A}}$: $[2^s - s - 2, s, 2^{s-1} - 2]$ anticode $\mathcal{A}_{s;[2,s-1]}, s \leq m - 1$

Optimality of C_{II}:
- For $s \in \{3,4,5\}$ is CM-optimal
- For **all** s is Griesmer-optimal

Anticode #3

- Let $\mathcal{A}_{m-1}$ be an anticode with the generator matrix

$$G_{\mathcal{A}} = \begin{pmatrix} 1 & 000 \ldots 00 \\ 0 & \\ \vdots & G_{m-1} \\ 0 & \end{pmatrix},\ G_{m-1} \text{ is the generator matrix of } S_{m-1}$$

- $\mathcal{A}_{m-1}$ is a $[2^{m-1}, m-1, 2^{m-2}+1]$ anticode

Parameters of Code C_{III}

- Let $\mathcal{A}_{m-1}$ be an anticode with the generator matrix

$$G_{\mathcal{A}} = \begin{pmatrix} 1 & 000\ldots00 \\ 0 & \\ \vdots & G_{m-1} \\ 0 & \end{pmatrix}$$, G_{m-1} is the generator matrix of S_{m-1}

- $\mathcal{A}_{m-1}$ is a $[2^{m-1}, m-1, 2^{m-2}+1]$ anticode
- **Theorem 6.** $G_{III} = G_m \setminus G_{\mathcal{A}} = \begin{pmatrix} 111\ldots11 \\ G_{m-1} \end{pmatrix}$ generates an

$$[2^{m-1}-1, m, 2^{m-2}-1]$$

$(3, t)$-LRC C_{III} with locality 3 and availability

$$t = \begin{cases} (2^{m-1}-4)/3 & \text{for odd } m \\ (2^{m-1}-5)/3 & \text{for even } m \end{cases}$$

Parameters of Code C_{III}

- Let $\mathcal{A}_{m-1}$ be an anticode with the generator matrix

$$G_{\mathcal{A}} = \begin{pmatrix} 1 & 000\dots00 \\ 0 & \\ \vdots & G_{m-1} \\ 0 & \end{pmatrix},$$ G_{m-1} is the generator matrix of S_{m-1}

- $\mathcal{A}_{m-1}$ is a $[2^{m-1}, m-1, 2^{m-2}+1]$ anticode
- **Theorem 6.** $G_{III} = G_m \setminus G_{\mathcal{A}} = \begin{pmatrix} 111\dots11 \\ G_{m-1} \end{pmatrix}$ generates an

$$[2^{m-1}-1, m, 2^{m-2}-1]$$

$(3,t)$-LRC C_{III} with locality 3 and availability

$$t = \begin{cases} (2^{m-1}-4)/3 & \text{for odd } m \\ (2^{m-1}-5)/3 & \text{for even } m \end{cases}$$

Size of a spread without one element in $\mathbb{F}_2^{m-1}$

Size of the largest partial spread in $\mathbb{F}_2^{m-1}$

Optimality of C_{III}

- Let $\mathcal{A}_{m-1}$ be an anticode with the generator matrix

$$G_{\mathcal{A}} = \begin{pmatrix} 1 & 000\dots00 \\ 0 & \\ \vdots & G_{m-1} \\ 0 & \end{pmatrix},$$ G_{m-1} is the generator matrix of S_{m-1}

- $\mathcal{A}_{m-1}$ is a $[2^{m-1}, m-1, 2^{m-2}+1]$ anticode
- <u>**Theorem 6.**</u> $G_{III} = G_m \setminus G_{\mathcal{A}} = \begin{pmatrix} 111\dots11 \\ G_{m-1} \end{pmatrix}$ generates an $[2^{m-1}-1, m, 2^{m-2}-1]$

$(3, t)$-LRC C_{III}

Optimality of C_{III}:

- For **all** s is CM-optimal
- For **all** s is Griesmer-optimal

Anticode #4

- Let $\mathcal{A}_s$ be an anticode with the generator matrix $G_{\mathcal{A}} = G_s$, the generator matrix of S_s
- $\mathcal{A}_s$ is a $[2^s - 1, s, 2^{s-1}]$ anticode

Parameters of Code C_{IV}

- Let $\mathcal{A}_s$ be an anticode with the generator matrix $G_{\mathcal{A}} = G_s$, the generator matrix of S_s
- $\mathcal{A}_s$ is a $[2^s - 1, s, 2^{s-1}]$ anticode
- **Theorem 7.** $G_{IV} = G_m \setminus G_s, s \leq m-1,$ generates an

$$[2^m - 2^s, m, 2^{m-1} - 2^{s-1}]$$

(r,t)-LRC C_{IV} with locality r and availability t given by

$$r = \begin{cases} 2 & \text{if } 2 \leq s \leq m-2 \\ 3 & \text{if } s = m-1 \end{cases}$$

$$t = \begin{cases} (2^{m-1} - 1)/3 & \text{if } s = m-1 \text{ and } m \text{ is odd} \\ (2^{m-1} - 5)/3 & \text{if } s = m-1 \text{ and } m \text{ is even} \\ 2^{m-1} - 2^s & \text{if } 2 \leq s \leq m-2 \end{cases}$$

Optimality of C_{IV}

- Let $\mathcal{A}_s$ be an anticode with the generator matrix $G_{\mathcal{A}} = G_s$, the generator matrix of S_s
- $\mathcal{A}_s$ is a $[2^s - 1, s, 2^{s-1}]$ anticode
- **<u>Theorem 7.</u>** $G_{IV} = G_m \setminus G_s, s \leq m-1,$ generates an

$$[2^m - 2^s, m, 2^{m-1} - 2^{s-1}]$$

(r,t)-LRC C_{IV} with locality r and availability t given by

$$r = \begin{cases} 2 & \text{if } 2 \leq s \leq m-2 \\ 3 & \text{if } s = m-1 \end{cases}$$

Optimality of C_{IV}:

- For **all** s is CM-optimal
- For **all** s is Griesmer-optimal

Summary

Ref.	[n, k, d]	Locality r	Availability t
$\mathcal{C}_{\mathrm{I}}$	$[2^m - \binom{s}{2} - 1, m, 2^{m-1} - \lfloor s^2/4 \rfloor]$	2	$2^{m-1} - \binom{s}{2} - 1$
$\mathcal{C}_{\mathrm{II}}$	$[2^m - 2^s + s + 1, m, 2^{m-1} - 2^{s-1} + 2]$	2	$2^{m-1} - 2^s + s + 1$
$\mathcal{C}_{\mathrm{III}}$	$[2^{m-1} - 1, m, 2^{m-2} - 1]$	3	$\begin{cases} \frac{2^{m-1}-4}{3} & \text{if } m \text{ odd}, \\ \frac{2^{m-1}-5}{3} & \text{if } m \text{ even}. \end{cases}$
$\mathcal{C}_{\mathrm{IV}}$	$[2^m - 2^s, m, 2^{m-1} - 2^{s-1}]$	$\begin{cases} 3 \text{ if } s = m-1, \\ 2 \text{ if } s \in [2, m-2] \end{cases}$	$\begin{cases} \frac{2^{m-1}-1}{3} & \text{if } s = m-1,\ m \text{ odd}, \\ \frac{2^{m-1}-5}{3} & \text{if } s = m-1,\ m \text{ even}, \\ 2^{m-1} - 2^s & \text{if } s \in [2, m-2]. \end{cases}$

Ref.	Optimal	
	Griesmer	CM
$\mathcal{C}_{\mathrm{I}}$	✓, for $s = 3, 4, 5$	✓, for $s = 3, 5$
$\mathcal{C}_{\mathrm{II}}$	✓, for $s = 3, 4, 5$	✓
$\mathcal{C}_{\mathrm{III}}$	✓	✓
$\mathcal{C}_{\mathrm{IV}}$	✓	✓

Some Numerical Examples

Reference	[n, k, d]	r	t	Optimal		Parameters	
				G.	CM	m	s
$\mathcal{C}_{\mathrm{I}}$, Thm. 4	$[28, 5, 14]$	2	12	✓	✓	5	3
	$[25, 5, 12]$	2	9		✓	5	4
	$[21, 5, 10]$	2	5	✓	✓	5	5
	$[60, 6, 30]$	2	28	✓	✓	6	3
	$[57, 6, 28]$	2	25		✓	6	4
	$[53, 6, 26]$	2	21	✓	✓	6	5
$\mathcal{C}_{\mathrm{II}}$, Thm. 5	$[21, 5, 10]$	2	5	✓	✓	5	4
	$[38, 6, 18]$	2	6	✓	✓	6	5
$\mathcal{C}_{\mathrm{III}}$, Thm. 6	$[31, 6, 15]$	3	9	✓	✓	6	—
	$[63, 7, 31]$	3	20	✓	✓	7	—
$\mathcal{C}_{\mathrm{IV}}$, Thm. 7	$[24, 5, 12]$	2	8	✓	✓	5	3
	$[48, 6, 24]$	2	16	✓	✓	6	4
	$[56, 6, 28]$	2	16	✓	✓	6	3

Outlook

- Constructions for binary $C_I, C_{II}, C_{III}, C_{IV}$ can be generalized for any field $\mathbb{F}_q$.
  - For $q \geq 3$, locality is always 2

Outlook

- Constructions for binary $C_I, C_{II}, C_{III}, C_{IV}$ can be generalized for any field $\mathbb{F}_q$.
  - For $q \geq 3$, locality is always 2

- The symbols of codes C_I, C_{II} have 2 (or 3 in some cases) different availabilities.
  - Derive tighter bounds for codes with different availabilities


Thank you!