Normalized tiling in $\mathbb{Z}_p$

Kristijan Tabak
Rochester Institute of Technology
kxtcad@rit.edu

NETCODE 16, April 2016

This work has been fully supported by Croatian Science Foundation under the project 1637 and COST action IC 1104

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$$D_i D_i^{(-1)} = (k-\lambda)\cdot 1_G + \lambda G,\ \ i \in [t],$$

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Let's provide one example of tiling in $\mathbb{Z}_{31}$. Then for

$$X_1 = \{1,5,11,24,25,27\}, X_2 = \{2,10,17,19,22,23\},$$
$$X_3 = \{3,4,7,13,15,20\},$$
$$X_4 = \{6,8,9,14,23,30\},\ X_5 = \{12,16,18,21,28,29\}.$$

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$$X_1 + X_2 + X_3 + X_4 + X_5 = \mathbb{Z}_{31} - 0_{\mathbb{Z}_{31}}$$

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Let $G = \langle a \rangle \cong \mathbb{Z}_p$ and $D \in G(p,k,1)_{DS}$ and $p > 3$. Then, there is $X \in G(p,k,1)_{DS}$ such that $\prod_{x \in X} x = 1$.

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Corollary

(fixed ⇒ normalized) *Let $D \in G(p,k,1)_{DS}$, $p \in \Pi$ and $t \in [p-1]$. If $D^{(t)} = D$, then D is normalized.*

Proposition

Let $G = \langle a \rangle \cong \mathbb{Z}_p$ *and* $p \in \Pi$. *Let* $\{D_i\}_1^t$ *is* $(p, k, 1)-$ *tiling. Then* $G = \sum_{i=1}^{t} D_i + 1$ *and*

1. $t = k - 1$,
2. *there is some* $M \in \mathbb{N}$ *such that* $4p + 3 = (2M + 1)^2$,
3. $k = M + 1$,
4. $\sum_{i=1}^{k} \chi_j(D_i) = -1$, $\chi_j \in Hom(G, \mathbb{C})$ *and* $\chi_j(a) = \varepsilon^j$, *where* $\varepsilon = e^{\frac{2\pi i}{p}}$, $j \in [p-1]$.
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Lemma

Let $G = \langle a \rangle \cong \mathbb{Z}_{31}$. *Then* $\{D_j\}_1^5$ *is a tiling where*

$D_j \in G(31,6,1)_{DS}$ *and* $D_j = \sum_{i=1}^{6} a^{\alpha_{ij}}$ *where*

$$\begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\\ \alpha_{i3}\\ \alpha_{i4}\\ \alpha_{i5}\end{bmatrix} = \begin{bmatrix} 1 & 5 & 11 & 24 & 25 & 27\\ 2 & 10 & 17 & 19 & 22 & 23\\ 3 & 4 & 7 & 13 & 15 & 20\\ 6 & 8 & 9 & 14 & 26 & 30\\ 12 & 16 & 18 & 21 & 28 & 29\end{bmatrix}$$

Additionaly, we have for $(\varphi, \psi, \theta) \in Aut(G)^3$ *given by*
$((\varphi(a), \psi(a), \theta(a))) = (a^{-1}, a^5, a^4)$ *and*
$(o(\varphi), o(\psi), o(\theta)) = (2,3,5)$.
Then $\varphi(D_j)$ *is spreded arround, and* $\psi(D_j) = D_j,\ j \in [5]$. *Also*
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Then $D_j = D_{j1} + D_{j2},\ j \in [5]$ and each D_{js} is normalized and for

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Proposition

Let $D \in G(31,6,1)_{DS}$ *and* $G = \langle a \rangle \cong \mathbb{Z}_{31}$. *Let* $\gamma \in Aut(G)$ *such that* $o(\gamma) = 5$ *and* $\{\gamma^i(D)\}$ *mutually disjoint. Then*

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Now, we introduce another notation.

$$TG(31,6,1) = \{\{D_j\}_1^5 \mid G = \sum_{j=1}^{5} D_j + 1,\ D_j \in G(31,6,1)_{DS}\},$$

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Let $X = \{D_j\}_1^5 \in TG(31,6,1)$ *in* $G = \langle a \rangle \cong \mathbb{Z}_{31}$*. If* $\mathbb{Z}_5 \hookrightarrow X$*, then* $X \in TG_n(31,6,1)$

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Since $(a^{\alpha_1})^{\theta^j} D_j \in Dev(D_j)$ and $a^{\alpha_j} D_j \in Dev(D_j)$, it means that they represent the same block of underling symmetric design, therefore representatives are equal. So, we get

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Thus, we can write

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$$B_0 = b_1 + b_1^{\psi} + b_1^{\psi^2} + b_2 + b_2^{\psi} + b_2^{\psi^2} = b_1^{\langle\psi\rangle} + b_2^{\langle\psi\rangle},$$
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$$D_0 = d_1 + d_1^{\psi} + d_1^{\psi^2} + d_2 + d_2^{\psi} + d_2^{\psi^2} = d_1^{\langle\psi\rangle} + d_2^{\langle\psi\rangle},$$
$$E_0 = e_1 + e_1^{\psi} + e_1^{\psi^2} + e_2 + e_2^{\psi} + e_2^{\psi^2} = e_1^{\langle\psi\rangle} + e_2^{\langle\psi\rangle},$$

Thus, we can write

$$A_0 = a_1 + a_1^{\psi} + a_1^{\psi^2} + a_2 + a_2^{\psi} + a_2^{\psi^2} = a_1^{\langle\psi\rangle} + a_2^{\langle\psi\rangle},$$
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Therefore, we can write

$$\begin{aligned} G^* &= a\left[a_1^{\langle\psi\rangle} + a_2^{\langle\psi\rangle}\right] + b\left[b_1^{\langle\psi\rangle} + b_2^{\langle\psi\rangle}\right] + \\ &+ c\left[c_1^{\langle\psi\rangle} + c_2^{\langle\psi\rangle}\right] + d\left[d_1^{\langle\psi\rangle} + d_2^{\langle\psi\rangle}\right] + \\ &= e\left[e_1^{\langle\psi\rangle} + e_2^{\langle\psi\rangle}\right] \end{aligned}$$

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Notice that

$$(a_1^{\langle\psi\rangle})^\theta = (a_1 + a_1^\psi + a_1^{\psi^2})^\theta = a_1^\theta + (a_1^\theta)^\psi + (a_1^\theta)^{\psi^2} = (a_1^\theta)^{\langle\psi\rangle}. \quad (1)$$

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Another important note is that from

$a[a_1^{\langle\psi\rangle} + a_2^{\langle\psi\rangle}] = b[b_1^{\langle\psi\rangle} + b_2^{\langle\psi\rangle}]$ we get $a^6 = b^6$, which means $a = b$. On the other hand, if $aa_1^{\langle\psi\rangle} + bb_1^{\langle\psi\rangle} = cc_1^{\langle\psi\rangle} + dd_1^{\langle\psi\rangle}$ we get $a^3b^3 = c^3d^3$, thus $ab = cd$.

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$a_1^{\langle\psi\rangle},\ a_2^{\langle\psi\rangle}, \ldots, e_1^{\langle\psi\rangle},\ e_2^{\langle\psi\rangle}$. Let us introduce notation

$$1_a \equiv aa_1^{\langle\psi\rangle},\ 2_a \equiv aa_2^{\langle\psi\rangle}, \ldots, 1_e \equiv ee_1^{\langle\psi\rangle},\ 2_e \equiv ee_2^{\langle\psi\rangle}.$$

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For example, if one orbit is $1_a 2_a 1_b 2_d 2_e$ it means

$$aa_1^{\langle\psi\rangle} \xrightarrow{\theta} aa_2^{\langle\psi\rangle},\ aa_2^{\langle\psi\rangle} \xrightarrow{\theta} bb_1^{\langle\psi\rangle},\ bb_1^{\langle\psi\rangle} \xrightarrow{\theta} dd_2^{\langle\psi\rangle},\ dd_2^{\langle\psi\rangle} \xrightarrow{\theta} ee_2^{\langle\psi\rangle}.$$

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$a^\theta = b,\ b^\theta = d,\ d^\theta = e,\ e^\theta = a$. Thus $a^{\theta^4} = a$, hence $a = 1$. Then also $b = d = e = 1$. From other orbit where we have c similarly we get $c = 1$. This means that every difference set is normalized.

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Then it means, up to some reordering $1_s \leftrightarrow 2_s$, where $s \in \{a, b, c, d, e\}$ we get θ acting in an orbit of length 5, and then by same approach as in Theorem $\mathbb{Z}_5 \hookrightarrow X$ we get $a = b = c = d = e = 1$. The same goes if θ has 5 or 10 fixed 'points'.

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Thank You!